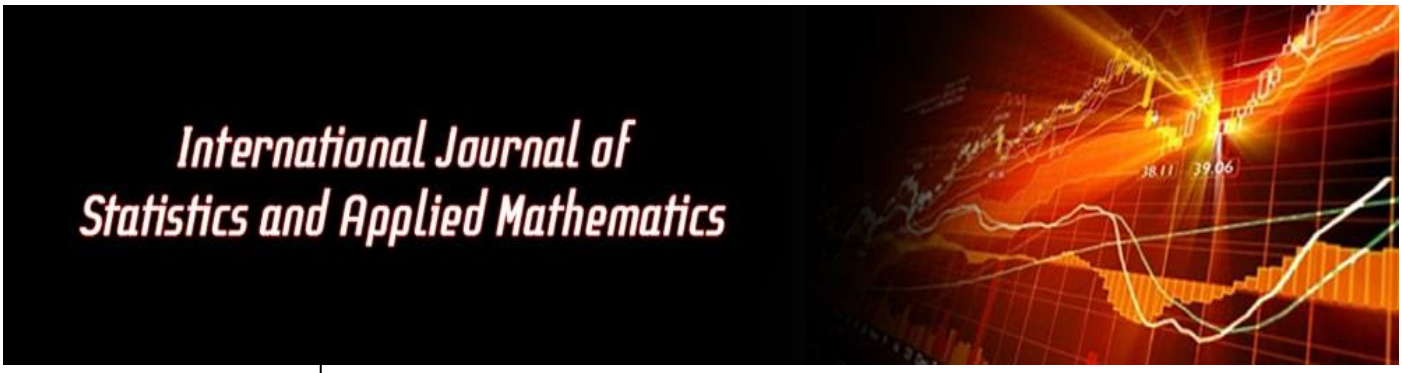

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Nonparametric tests for the umbrella alternative in a mixed design

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Abstract
Various tests are proposed to test for differences in treatment means assuming an umbrella alternative, if the means are different. The tests are for a mixed design consisting of a CRD, an RCBD and a BIBD. All tests considered have an asymptotic normal distribution under the null hypothesis. Powers are estimated for a variety of underlying distributions including normal, exponential, and t-distribution with 3 degrees of freedom. Three, four and five populations are considered. Different peak points are considered. Cases are considered in which the peak mean is the same as another mean and when it is distinct. An overall recommendation is given for one test statistic for all the underlying distributions.

Keywords: Completely randomized design, randomized complete block design, balanced incomplete block design, known peak, missing observations

1. Introduction
A nonparametric approach in testing hypotheses is sometimes used by researchers since fewer assumptions are required. In this particular research, we are concentrating on nonparametric tests for ordered alternatives for the means. In particular, we are considering tests for the umbrella alternative for means when the turning point, if it exists, can be assumed to be known a prior. This may occur when one is dealing with different dosages of the same drug for treatment of a disease. With increasing dosages of a drug, the treatment may benefit patients up to a point and then start to have decreasing effects, if there are any effects at all. The test of hypotheses for the umbrella alternative is given as follows

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k \text{ vs } H_a: \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{p-1} \leq \mu_p \geq \mu_{p+1} \geq \cdots \geq \mu_k$$

With at least one strict inequality where p is the peak of the umbrella and $\mu_1, \ldots, \mu_k$ are the location parameters of the populations with i=1, 2 k. A researcher must also decide on the type of experimental design they wish to use when considering hypothesis testing on means. It is possible that a researcher may start with one design and then must switch to a different design because of missing observations or because the cost of conducting the experiment becomes too high for the initial design. A researcher may start with a Randomized Complete Block Design and realize that this is too expensive of design to continue or they may have some missing observations within a block, or may not have enough experimental units within each block to have every treatment appearing in a block. The researcher at this point may switch to a completely randomized design (CRD) or balanced incomplete block design (BIBD). It would be a waste of time and resources for not using all available data. For this reason, we would need a test that combines three designs namely RCBD, BIBD and CRD.
Other researchers have been working on tests for mixed designs to use all the information from their data at hand. Dubnicka, Blair, and Hettmansperger (2002) [2] considered a design consisting of two independent samples and then paired samples in testing the equality of two location parameters. Magel, Terpstra, Canonizado, and Park (2010) [8] considered a test for the umbrella alternative with known peak, p, that was for a mixed design consisting of an RCBD and a CRD.

They considered equal sample sizes for treatments in CRD; and one observation for each treatment in each block in RCBD. Alvo and Cabilio (1995) [1] proposed a test for missing observations within a block which assigns an average of ranks appearing in a block to the missing observations. Magel and Ndungu (2011) [7] developed a test statistic under the umbrella alternative for a design consisting of both complete and incomplete blocks.

1.1 Esra and Fikri

Esra and Fikri (2016) [3] modified the Mack-Wolfe test in the CRD case. When the peak is known, their test statistic, $MW_{d,}$ is the sum of the weighted Mann-Whitney counts which consider how far apart the populations are from each other. The proposed statistic is the sum of two modified Jonckheere statistics as investigated by Neuhauser, Liu, and Horton (1988) [9] for ordered alternatives and is given by

$$MW_{\mathrm{d}} = \sum_{\mathrm{u}=1}^{\mathrm{v}-1}\sum_{\mathrm{v}=2}^{p}(\mathrm{v}-\mathrm{u})U_{uv} + \sum_{\mathrm{u}=p}^{\mathrm{v}-1}\sum_{\mathrm{p}=i+1}^{k}(\mathrm{u}-\mathrm{v})U_{vu}$$

Under the H_0 that all population means are equal, the expected value and variance of (MW_d) are respectively given below.

$$E_0(MW_{\mathrm{d}}) = \frac{n^2}{2}\left[\binom{p+1}{3} + \binom{k-p+2}{3}\right]$$

and

$$var_0(MW_{\mathrm{d}}) = \frac{n^2p^2(p^2-1)(np+1) + n^2(k-p+1)((k-p+1)^2-1)(n(k-p+1)+1)}{144} + \frac{n^3p(p-1)(k-p)(k-p+1)}{24}$$

When the sample size gets large, the standard version MW_d^* has an asymptotic standard normal distribution and the null hypothesis is rejected for large values.

1.2 Magel, Tersptra, Canonizado, and Park (2010) [8]

Magel, Terpstra Jeff, Canonizado and Park (2010) [8] proposed two tests for a mixed design under the umbrella alternative. The mixed design consisted of a Completely Randomized Design (CRD) and Randomized Complete Block Design (RCBD). Their first proposed test is given by

A=$MW_d^* + KK^*$

Where MW_d^* and $KK^*$are the standardized versions of the usual Mack-Wolfe and Kim-Kim tests. Under H_0, both have an asymptotic standard normal distribution, thus the expected value and variance are given by

$$E_0(MW_d^*) + E_0(KK^*) = 0$$

and

$$var_0(MW_d^*) + var_0(KK^*) = 1 + 1 = 2$$

The standardized version of the first proposed test is denoted by A^*. It has an asymptotic standard normal distribution under the null and the null is rejected for large values.

The second proposed A_p^{***} test is given by

$$A_p^{***} = MW + KK$$

Where $MW$and KK are the unstandardized versions of the Mack-Wolfe and Kim-Kim (1992) [5] tests. Under H_0, the expected value and variance of A_p^{***} are the sum of the means and variances for the Mack-Wolfe and Kim-Kim tests (Mack-Wolfe (1981) [6] for the CRD Kim-Kim (1992) [5] for RCBD, respectively).

The standardized version of the second proposed test is denoted by A^{***} which has an asymptotic standard normal distribution under the null and the null is rejected for large values.

1.3 Modified Mack-Wolfe Test Statistics for RCRD (Distance and Distance Squared

We considered two modifications of the Kim-Kim (1992) [5] test for the RCBD. One modification considers the distances between each of the populations when comparing two of the k populations as in Esra and Fikri (2016) [3] and Neuhauser *et al*. (1998) [9] for the JT. The second modification considers the distances squared when comparing two of the k populations. The first and second modified tests are denoted by KK_d and KK_{ds} and are as follows.

$$KK_d = \sum_{i=1}^{b}\left\{\sum_{u=1}^{v-1}\sum_{v=2}^{p}(v-u)U_{iuv} + \sum_{u=p}^{v-1}\sum_{v=p+1}^{k}(u-v)U_{ivu}\right\}$$

Where the modified Mack and Wolfe (1981) [6] test statistic of the i^{th} block is calculated, and then summed up over the blocks with b blocks, p is the known peak, k is the number of treatments, U_{iuv} and U_{ivu} are the U statistics associated with the i^{th} block where U_{iuv} counts the number of times the observation receiving treatment u is less than the observation receiving treatment v
We are extending this weighting modification to the umbrella alternative. Under H_0, the test statistic will have an asymptotic normal distribution (follows from Kim and Kim (1992)) [5] where KK_d^* is the standardized version. The value of KK_d^* will be large whenever the alternative hypothesis is true.
We propose a distance squared modification of the Kim-Kim (1992) [5] test for the randomized complete block design as follows:

$$KK_{ds} = \sum_{i=1}^{b}\left\{\sum_{u=1}^{v-1}\sum_{v=2}^{p}(v-u)^2U_{iuv} + \sum_{u=p}^{v-1}\sum_{v=p+1}^{k}(u-v)^2U_{ivu}\right\}$$

Where the modified Mack-Wolfe (1981) [6] test statistic of the i^{th} block is calculated for every block and then summed up over all the blocks. KK_{ds}^* is the standardized version and follows an asymptotic standard normal when the null hypothesis is true. The null hypothesis will be rejected for large values.

1.5 Asare and Magel Modified Distance Test Statistics for BIBD

We are introducing a modification to the Magel and Ndungu (2011) [7] test statistics for the BIBD. The modified test statistics give the weight $(j-i)$ to U_{ijb} in two forms; where there are no missing observations and where there are missing observations. Magel and Ndungu (2011) [7] gave the test statistic as follows:

$$M = \sum_{b=1}^{n} M_b$$

Where,

$$M_b = \sum_{i<j}\sum_{j+1>p} U_{ijb}$$

U_{ijb} is the number of pairs of observations (x, y) in block b for which $x < y$ if $x \,\&\, y < peak$. y can be the peak $x > y$ if $x \,\&\, y > peak$, x can be the peak. Treatments are not compared across the peak. Without a loss of generality, assume a pair $i < j \leq peak$ where j can also be the peak: If there are no missing observations, then $U_{ijb} = 0$ if i and j are greater than the peak p. $(j-i)$ if i and j are less than the peak and according to the positions of i^{th} and j^{th}in the populations. The further i and j are from the peak p (j can be the peak), the greater the weight. If there are missing observations, then $\frac{1}{2} * (j-i)$ if i and j are missing. $(j-i) * \left[1 - \frac{r_i}{k+1}\right]$ if j is missing and i is not, where r_i is the rank of i within the block and k is the number of treatments appearing in the block. $(j-i) * \frac{r_j}{k+1}$ if i is missing and j is not. We denote the Distance Modification Mungai Test statistic as M_d where this is calculated within each block and then summed up over the blocks. The mean and variance for this can be found by summing up the means and variances over all the blocks. Each mean and variance will depend on the number of observations in the block, the assumed peak, and where we have missing observations.
An example is given in Table 1 for calculating the expected value and variance of block b for 5 populations, with 3 observations for each block and the observations missing for the 4th and 5th treatment with a known umbrella point of 2. Step 1 shows all the different arrangements possible for 3 observations with 1, 2, and 3 denoting the ranks of the observations. Step 2 gives the value of the test statistic for each arrangement. Step 3 gives the mean for when the fourth and fifth observations are missing and Step 4 calculates the variance for that situation.

Table 1: t=5, k=3, Peak=2 & Fourth & Fifth Observations Missing

Step 1					Step 2	Step 3	Step 4
1	2	3	-	-	25/4	11/2	21/8
1	3	2	-	-	31/4		
2	1	3	-	-	16/4		
2	3	1	-	-	28/4		
3	1	2	-	-	13/4		
3	2	1	-	-	19/4		

The mean and variance for this situation are calculated

- **Step 2:** Using the first arrangement

$(1, 2) \rightarrow U_{12a} = 1; (2, 3) \rightarrow U_{23a} = 0; (2, \_) \rightarrow U_{24a} = \frac{4}{4} = 1;$

$(2, \_) \rightarrow U_{25a} = 3 * \left(\frac{2}{4}\right) = \frac{6}{4}; (3, \_) \rightarrow U_{34a} = \frac{3}{4}, (3, \_) \rightarrow U_{35a} = 2 * \frac{3}{4} = \frac{6}{4}$

$(\_, \_) \rightarrow U_{34a} = \frac{1}{2}.$

$$\sum U_{ija} = 1 + 0 + 1 + \frac{6}{4} + \frac{3}{4} + \frac{6}{4} + \frac{1}{2} = \frac{25}{4}$$

$E(M_d) =$

$$= \frac{\frac{25}{4} + \frac{31}{4} + \cdots + \frac{13}{4} + \frac{19}{4}}{6} = \frac{11}{2}$$

Step 4: $\sigma_d^2 =$

$$= \frac{(\frac{25}{4} - 5.5)^2 + \left(\frac{31}{4} - 5.5\right)^2 + \cdots + \left(\frac{19}{4} - 5.5\right)^2}{6} = \frac{21}{8}$$

The standardized version of the Distance Modification Mungai is denoted by M_d^* and the null hypothesis is rejected for large values.

1.6 Asare and Magel Modified Squared Distance Test Statistics for BIBD

Similarly, we are introducing a Squared Distance Modification test to the Magel and Ndungu (2011) [7] test for complete and incomplete blocks design. The modified test gives each whether there are missing observations or not If there are no missing observations, then $U_{ijb} = 0$ if i and j are greater than the peak p; $(j - i)^2$ if i and j are less than the peak p. If there are missing observations, then $\frac{1}{2} * (j - i)^2$ if i and j are missing; $(j - i)^2 * \left[1 - \frac{r_i}{k+1}\right]$ if j is missing and i is not where r_i is the rank of i within the block and k is the number of treatments appearing in the block; $(j - i)^2 * \frac{r_j}{k+1}$ if i is missing and j is not. The standardized Modified Squared Mungai statistic is denoted by M_{ds}^* and is rejected for large values.

2. Proposed Test for a Mixed Design of RCBD, BIBD and CRD

We are proposing three versions of nonparametric test statistics for the mixed design consisting of an RCBD, a BIBD, and a CRD. The first version just adds the original versions of the Mack-Wolfe (1981) [3], Kim-Kim (1992) [5], and Magel-Ndungu (2011) [7]. The second version using the distance modifications of each of the test statistics. The third version uses the distance squared modifications of each of the test statistics. For each of the three versions, we developed two test statistics. In the first test statistic, we first standardized each of the aforementioned test statistics, added them together and divided by square root of 3. In the second set of test statistics, we added the unstandardized versions of the aforementioned test statistics together and then standardized the resulting test statistic. Results of these standardized tests were compared to each other to see which test statistic performs better as far as estimated power.

2.1. Proposed Test One: Non-Modification for Three Mixed Designs: Mack-Wolfe (1981) [3] for CRD, Kim-Kim (1992) [5] for RCBD and Magel and Ndungu (2011) [7] for (BIBD)

Standardized First: The first proposed test is given by

$$T = MW^* + KK^* + M^*$$

Where MW^*, KK^* M^* are the standardized versions of the Mack-Wolfe, Kim-Kim and Magel and Ndungu tests with the mean being 0 with variance equal to 3., The standardized version of the first proposed test is given by

$$T^* = \frac{T - E_0(T)}{\sqrt{var_0(T)}} = \frac{T - 0}{\sqrt{3}}$$

Under H_0, T^* has an asymptotic standard normal distribution. The null is rejected for large values.

Standardized Last: The second proposed test is given by

$$T_2 = MW + KK + M$$

The unstandardized versions of the tests are added together and then standardized and denoted by T_2^*.
Under H_0, T_2^* has an asymptotic standard normal distribution. The null is rejected for large values.

2.2. Proposed Test Two: Distance Modification for Three Designs Mack-Wolfe (1981) [3] for CRD, Kim-Kim (1992) [5] for RCBD and Magel and Ndungu (2011) [7] for BIBD

Standardized First: The third proposed test, KA_{mp} is given by

$$T_d = MW_d^* + KK_d{}^* + M_d^*$$

Where, $MW_d^*\ KK_d{}^* M_d^*$ are the standardized distance modifications of the Mack-Wolfe (1981) [3] Kim-Kim (1992) [5] and Magel and Ndungu (2011) [7] tests. The standardized version of the third proposed test is given by

$$T_d^* = \frac{T_d - E_0(T_d)}{\sqrt{var_0(T_d)}} = \frac{T_d - 0}{\sqrt{3}}$$

Under H_0, T_d^* has an asymptotic standard normal distribution. The null is rejected for large values.

Standardized last: The fourth proposed KA_{mp}^* test is given by

$$T_{d2} = MW_d + KK_d + M_d$$

The unstandardized distance modifications of the tests are added together and then that test statistic is standardized. The standardized version of the fourth proposed test is given by T_{d2}^* and is rejected for large values.

2.3 Proposed Test Three: Squared Distance Modification for three designs (Mack-Wolfe (1981) [3] for CRD, Kim-Kim (1992) [5] for RCBD and Magel-Ndungu (2011) [7] for BIBD)

Standardized First: The fifth proposed test $A_{mp,}^{**}$ is given by

$$T_{ds} = MW_{ds}^* + KK_{ds}^* + M_{ds}^*$$

Where $MW_{ds}^*\ KK_{ds}^*\ M_s^*$ are the standardized versions of the Distance Squared-Modifications of the Mack-Wolfe (1981) [3,], Kim-Kim (1992) [5] and Magel-Ndungu (2011) [7]. The standardized version of the fifth proposed test is denoted by T_{ds}^* and has an asymptotic standard normal distribution. The null is rejected for large values.

Standardized last: The sixth proposed A_{mp}^{**} test is given by

$$T_{ds2} = MW_{ds} + KA_{ds} + M_{ds}$$

Where the unstandardized Distance Squared-Modifications of the tests are added together. The standardized last version of the test statistic is denoted by $T_{ds2}^*$and is rejected for large values.

3. Simulation Study

A simulation study was implemented using SAS version 9.4. It was conducted to estimate the type one errors and the powers of the proposed test statistics. In this research, the data are generated from a mixed design consisting of portions of an RCBD, a CRD and a BIBD. The function RAND found in the software was used to simulate observations. Three different underlying distributions were considered including the normal, exponential and t distribution with three degrees of freedom. Missing observations were created using the Uniform distribution. Simulated observations were individually assigned a probability p of missing by using the call function RAND ('Uniform'). The Uniform distribution produced a random number between zero and one. Three probabilities of missing values were considered.

The alpha values were estimated for all situations considered based on the number of times the null hypothesis was rejected in 5000 simulations. These were compared to the stated alpha values which were always 0.05 for this study. The powers of each of the tests were then estimated under a variety of conditions based on 5000 simulations for each condition. In this paper, three, four and five treatments are considered with the assumption that the turning point is known.

4. Sample Sizes Considered for a Mixed Design of BIBD, CRD and RCBD

For a mixed design of BIBD, CRD and RCBD, the following combinations of sample size to the number of blocks were considered for three and four populations. In all cases studied, the number of blocks for the RCBD was equal to the number of blocks for the BIBD. Cases were considered when the CRD had the same sample size as the number of blocks (5,5.5), where the number of blocks was greater than the sample size for the CRD (10,5,10) and (15,5,15), and when the sample size for the CRD was higher than the number of blocks (5,10,5) and (5,15,5)

5. Configuration of Parameters and peaks considered in the simulation study

We considered 3 populations with peak at 2, 4 populations with peak at 2 and 3, and 5 populations with peak at 2, 3, or 4. Various configurations of means were considered in our simulation study. These included cases when the peak was distinct and cases when the peak was not distinct

6. Mixed Design of RCBD, BIBD and CRD Results

The results are given for 5 treatments with the peak at 2, 3, or 4. Results for 3 and 4 treatments are similar. The results were similar for the t distribution with 3 degrees of freedom and for the normal distribution as far as which test statistic is better. Only the results from the normal and exponential distributions are given and for sample sizes of (5, 5, 5).

Table 2: Treatments=5, BIBD=5, CRD=5, RCBD=5 Peak=2 (Two Missing Observations)

Distribution	Location	Non-modification		Distance-Modification		Distance Squared-Modification	
	μ_1 μ_2 μ_3 μ_4 μ_5	First	Last	First	Last	First	Last
Normal	0 0 0 0 0	0.0486	0.0528	0.0512	0.0520	0.0438	0.0474
	1 1 0.5 0.5 0.5	0.2062	0.1712	0.2240	0.1994	0.2228	0.2116
	0.5 1 1 1 0.7	0.1960	0.1652	0.1870	0.1820	0.1832	0.1818
	0.7 1 0.7 0.7 0.5	0.2742	0.2252	0.2760	0.2428	0.2656	0.2550
	1 1 0.5 0.5 0.2	0.3874	0.3358	0.4112	0.3672	0.4032	0.3896
	0.75 1 0.75 0.5 0.25	0.4752	0.3956	0.4762	0.4296	0.4722	0.4566
	1 1 1 1 0	0.4994	0.4078	0.5164	0.4608	0.5238	0.4914
	1 1 0.5 0.2 0	0.6054	0.4848	0.6222	0.5658	0.6090	0.5932
	0 1.6 0.8 0.4 0.2	0.9654	0.9174	0.9636	0.9442	0.9502	0.9430
Exponential	0 0 0 0 0	0.0430	0.0480	0.0496	0.0540	0.0540	0.0502
	1 1 0.5 0.5 0.5	0.3316	0.2604	0.3544	0.2946	0.3308	0.3244
	0.5 1 1 1 0.7	0.3554	0.2790	0.3200	0.2968	0.2958	0.2862
	0.7 1 0.7 0.7 0.5	0.4746	0.3894	0.4874	0.4350	0.4530	0.4510
	1 1 0.5 0.5 0.2	0.6136	0.5110	0.6540	0.5770	0.6350	0.6166
	0.75 1 0.75 0.5 0.25	0.7622	0.6540	0.7640	0.7024	0.7322	0.7276
	1 1 1 1 0	0.6400	0.5310	0.6770	0.5942	0.6802	0.6620
	1 1 0.5 0.2 0	0.8340	0.7248	0.8490	0.7950	0.8360	0.8118
	0 1.6 0.8 0.4 0.2	0.9984	0.9870	0.9950	0.9944	0.9936	0.9890

Table 3: Treatments=5, BIBD=5, CRD=5, RCBD=5 Peak=3 (Two Missing Observations)

Distribution	Location	Non-modification		Distance-Modification		Distance Squared-Modification	
	μ_1 μ_2 μ_3 μ_4 μ_5	First	Last	First	Last	First	Last
Normal	0 0 0 0 0	0.0484	0.0460	0.0534	0.0548	0.0496	0.0530
	0 0.5 1 1 0.7	0.3054	0.2974	0.4894	0.4130	0.4920	0.4434
	0.7 0.7 1 0.5 0.5	0.2096	0.2024	0.2540	0.2224	0.2758	0.2398
	0 0.5 1 0.5 0	0.6800	0.6400	0.8018	0.7194	0.8022	0.7852
	0 1 1 0 0	0.8204	0.6860	0.7912	0.7132	0.7958	0.7508
	1 1 1 0.5 0	0.4340	0.3078	0.3324	0.2760	0.3276	0.3106
	1 1 1 1 0	0.3750	0.2698	0.3222	0.2618	0.3310	0.2836
Exponential	0 0 0 0 0	0.0492	0.0474	0.0480	0.0484	0.0590	0.0490
	0 0.5 1 1 0.7	0.4712	0.4648	0.7276	0.6454	0.7284	0.6894
	0.7 0.7 1 0.5 0.5	0.3476	0.3078	0.4336	0.3558	0.4346	0.4000
	0 0.5 1 0.5 0	0.8858	0.8746	0.9686	0.9312	0.9696	0.9576
	0 1 1 0 0	0.9476	0.8636	0.9374	0.8798	0.9366	0.9152
	1 1 1 0.5 0	0.6302	0.4398	0.4662	0.3994	0.4758	0.4424
	1 1 1 1 0	0.4982	0.3672	0.4078	0.3468	0.4030	0.3838

Table 4: Treatments=5, BIBD=5, CRD=5, RCBD=5 Peak=4 (Two Missing Observations)

Distribution	Location	Non-modification		Distance-Modification		Distance Squared-Modification	
	μ_1 μ_2 μ_3 μ_4 μ_5	First	Last	First	Last	First	Last
Normal	0 0 0 0 0	0.0480	0.0460	0.0468	0.0466	0.0582	0.0508
	0.5 0.5 0.5 1 1	0.2180	0.1708	0.2218	0.2030	0.1962	0.1818
	0 0.5 1 1 0.7	0.6946	0.5878	0.7120	0.6682	0.6322	0.6070
	0 0.5 1 1 1	0.6336	0.5296	0.6508	0.5892	0.5352	0.4922
	0 0 1 1 1	0.7312	0.5974	0.7564	0.6858	0.5868	0.5676
	0.5 0.5 0.7 1 0.7	0.3164	0.2630	0.3190	0.2722	0.2064	0.2118
	1 1 1 1 0	0.1234	0.1074	0.1252	0.1062	0.1900	0.1630
	1 0.4 0.8 1.6 0.4	0.9794	0.9272	0.9748	0.9518	0.8832	0.8786
	0 0.25 0.5 1 0.25	0.7516	0.6282	0.7532	0.6914	0.5578	0.5476
Exponential	0 0 0 0 0	0.0514	0.0528	0.0494	0.0486	0.0534	0.0492
	0.5 0.5 0.5 1 1	0.3274	0.2462	0.3490	0.2902	0.2768	0.2572
	0 0.5 1 1 0.7	0.9028	0.8004	0.9126	0.8454	0.8506	0.8172
	0 0.5 1 1 1	0.8334	0.7356	0.8606	0.8018	0.7280	0.7082
	0 0 1 1 1	0.8724	0.7714	0.8950	0.8440	0.7504	0.7244
	0.5 0.5 0.7 1 0.7	0.5502	0.4398	0.5446	0.4832	0.3424	0.3524
	1 1 1 1 0	0.1474	0.1172	0.1284	0.1070	0.2288	0.1784
	1 0.4 0.8 1.6 0.4	0.9982	0.9922	0.9976	0.9964	0.9800	0.9832
	0 0.25 0.5 1 0.25	0.9448	0.8806	0.9334	0.9144	0.8206	0.8152

6.1 Five treatment at peak two

Overall, the standardize first versions of the tests are better. There are times when the non-modification version has the highest estimated power, times when the distance-modification does, and times when the distance-squared modification does. However, when one closely examines all of the estimated powers, the distance-modification version has powers that are the highest or powers that are in the middle and close to the test statistic with the highest estimated powers. Since the configuration of parameters would be unknown ahead of time, the distance-modification test statistic in which you standardize first is recommended.

6.2 Five treatments at peak four

With five treatments at peak three or four, the standardized first provides estimated powers slightly higher than the standardized last versions. This was true regardless of the underlying distribution and the proportion of BIBD to CRD and RCBD. The distance modification had the highest estimated powers of the test statistics, except where two population parameters are equal and different from the peak and the other parameters are different from each other, such as (0.0,0.25,0.5,1.0,0.25). In this case, the non-modification test performed better. Where three population parameters are equal to the peak and the other parameter different from the peak, such as (1.0,1.0,1.0,1.0,0.0), the distance squared modification performed better. Overall, we recommend using the distance modification since this ended up usually having the highest estimated power or had an estimated power between the other two test statistics and the exact configuration would be unknown ahead of time since we are testing to see if there is even a difference.

7. Conclusion and Discussion

With three treatments at peak two, the standardized first versions of the test statistics generally performed better than the standardized last version for the conditions considered. This was also true for four treatments at peaks two and three or five treatments with peak at 2, 3, or 4. The results among the three distinct modification tests vary from one configuration to the other and from one distribution to the other. We are overall recommending the distance modification since it generally had the highest or second highest powers for the configurations considered.

8. References

1. Alvo M, Cabilio P. Testing ordered alternatives in the presence of incomplete data. Journal of the American Statistical Association. 1995;90(431):1015-1024.
2. Dubnicka S, Blair C, Hettmansperger T. Ranked-based procedures for mixed paired and two-sample designs. Journal of Modern Applied Statistics Methods. 2002;1(1):32-41.
3. Esra G, Fikri G. A modified Mack-Wolfe test for the umbrella alternative problem. Communications in Statistics-Theory and Methods. 2016;45(24):7226-7241.
4. Jonckheere AR. A distribution-free k-sample test against ordered alternatives. Biometrika. 1954;41(1-2):133-145.
5. Kim DH, Kim YC. Distribution-free tests for umbrella alternatives in a randomized block design. Journal of Nonparametric Statistics. 1992;1(4):277-285.
6. Mack GA, Wolfe DA. K-sample rank tests for umbrella alternatives. Journal of the American Statistical Association. 1981;76(373):175-181.
7. Magel R, Ndungu A. A nonparametric test for the non-decreasing alternative in an incomplete block design. Fargo (ND): Department of Statistics, North Dakota State University, 2011.
8. Magel R, Terpstra J, Canonizado K, Park J. Nonparametric tests for mixed designs. Communications in Statistics-Simulation and Computation. 2010;39(6):1228-1250.
9. Neuhauser M, Liu PY, Horton L. Nonparametric tests for trend: Jonckheere's test, a modification, and a maximum test. Biometrical Journal. 1998;40(8):899-909.