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Generalized modular and orlicz-paranormed sequence spaces and their matrix domains

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Abstract
We develop a unified modular framework for Orlicz-type sequence spaces generated by either a single Orlicz function or a coordinatewise family of Orlicz functions. Under a uniform Δ_2 growth hypothesis we prove that the associated Luxemburg functional induces a complete BK-topology. We then introduce Maddox-type variable-exponent Orlicz spaces $\ell_\psi(p)$ and show that they inherit the same BK-structure. Finally, we study Orlicz-paranormed matrix domains $\ell_\psi(A)$ under lower triangular matrices with nonzero diagonal, establishing completeness, inclusion criteria driven by global dominance of Orlicz functions and by weight monotonicity, explicit descriptions of the Kothe-Toeplitz α-, β-, and γ-duals in terms of the conjugate Orlicz function, and concrete Schauder bases transported via A^{-1}.

Keywords: Orlicz sequence spaces, modular sequence spaces, Luxemburg functional, Δ_2 condition, Maddox-type spaces, variable exponent sequence spaces, BK-spaces, matrix domains, Norlund matrices, Kothe-Toeplitz duals, Schauder basis

1. Introduction
Orlicz sequence spaces and their modular generalizations provide a flexible replacement for classical ℓ_p scales, allowing one to model non-power growth, weighted tail control, and variable local behavior in a single functional-analytic setting. The modular approach (via Luxemburg type functionals) is particularly well adapted to matrix transformations and to the study of K¨othe-Toeplitz duals; see, for example, Musielak [2] and Woo [1]. The purpose of this paper is threefold:

1. To present a concise, self-contained BK-space development for modular sequence spaces generated by a family of Orlicz functions under a uniform Δ_2 condition;
2. To introduce and formalize a Maddox-type Orlicz construction $\ell_\psi(p)$ (variable exponent in the Orlicz input) and record its basic structural consequences;
3. To transfer these results to Orlicz-paranormed matrix domains $\ell_\psi(A)$ associated with lower triangular matrices (including Norlund-type matrices), and to derive dual and basis representations in terms of ψ^* and A^{-1}.

Throughout, we focus on statements in theorem form and omit repetitive parallel arguments, referring to standard modular-space theory when appropriate.

2. Preliminaries: Orlicz functions, modulars, and Luxemburg functionals
Orlicz functions and conjugates
Definition 2.1 (Orlicz function)**.** A function ψ: $[0,\infty) \to [0,\infty)$ is an Orlicz function if:

1. $\psi(0) = 0$ and $\psi(u) > 0$ for all u > 0;
2. ψ is nondecreasing and convex on $[0,\infty)$;
3. $\psi(u) \to\infty$ as $u \to\infty$.

Definition 2.2 (Conjugate Orlicz function)**.** The conjugate Orlicz function ψ^*: $[0,\infty) \to [0,\infty]$ is

$$\psi^*(v) := \sup_{u \geq 0}\ uv - \psi(u)\big), \qquad v \geq 0$$

A basic tool is Young's inequality (see, e.g., [2]):

$uv \leq \psi(u) + \psi^*(v)(u,v \geq 0)$. (1)

Modular sequence spaces
Let $\omega := \{x = (x_k)_{k\geq 0:}\ x_k \in C\}$ denote the space of all complex sequences.

Definition 2.3 (Family modular)**.** Let $\Psi = (\psi_k)_{k\geq 0}$ be a sequence of Orlicz functions. For $x \in \omega$ and $\mu > 0$ define the modular

$$I_\Psi(x;\mu) := \sum_{k=0}^{\infty} \psi_k\ \ \mu|x_k|\big) \in [0,\infty], \qquad (2)$$

and set $I_\Psi(x) := I_\Psi(x;1)$. The modular sequence space is

$$\ell(\Psi) := \big\{x \in \omega :\ I_\Psi(x;\mu) < \infty \text{ for some} \mu > 0\big\}. \qquad (3)$$

Definition 2.4 (Luxemburg functional)**.** For $x \in \ell(\Psi)$ define

$$q_\Psi(x) := \inf\big\{\lambda > 0 :\ I_\Psi(x/\lambda) \leq 1\big\}. \qquad (4)$$

Definition 2.5 (Uniform Δ_2 condition)**.** We say $\Psi = (\psi_k)$ satisfies a uniform Δ_2 condition if there exists $K \geq 1$ such that

$\psi_k(2u) \leq K\ \psi_k(u)\ (u \geq 0,\ k \geq 0)$. (5)

Main result I: BK-structure of modular Orlicz sequence spaces
We record the standard BK-space consequence of convex modular theory, specialized to the present family setting; see [1, 2, 3].

Theorem 3.1 (BK-property of $\ell(\Psi)$)**.** Assume $\Psi = (\psi_k)$ satisfies the uniform Δ_2 condition (5).
Then
1. q_Ψ is a paranorm on $\ell(\Psi)$ and $(\ell(\Psi), q_\Psi)$ is complete.
2. Each coordinate functional $p_k(x) = x_k$ is continuous on $\ell(\Psi)$.
3. Consequently, q_Ψ is equivalent to a norm on $\ell(\Psi)$; in particular, $\ell(\Psi)$ is a BK-space under this topology.

Proof sketch. Paranorm properties follow from convexity and monotonicity of each ψ_k, together with the Luxemburg definition (4); the triangle inequality is obtained by convexity after writing $(x + y)/(\lambda + \mu)$ as a convex combination of x/λ and y/μ. Completeness and continuity of coordinates are standard consequences of modular Cauchy criteria combined with (5); see [2, 1]. The final norm-equivalence is obtained by the usual renorming argument for complete K-spaces induced by convex modulars under Δ_2; cf. [2].

4. Orlicz and Maddox-type variable-exponent constructions
Classical Orlicz sequence spaces Fix a single Orlicz function ψ.
Definition 4.1 (Orlicz sequence space). Define

$$I_\psi(x;\mu) := \sum_{k=0}^{\infty} \psi\ \ \mu|x_k|\big), \qquad \ell_\psi := \{x \in \omega :\ I_\psi(x;\mu) < \infty \text{ for some } \mu > 0\},$$

and the Luxemburg functional

$$q_\psi(x) := \inf\big\{\lambda > 0 :\ I_\psi(x/\lambda;1) \leq 1\big\}.$$

Corollary 4.2. If ψ satisfies $\psi(2u) \leq K\psi(u)$ for all $u \geq 0$, then (ℓ_ψ, q_ψ) is a BK-space.
Proof. Apply Theorem 3.1 with $\psi_k \equiv \psi$.

Maddox-type Orlicz spaces $\ell_\psi(p)$

Definition 4.3 (Variable-exponent Orlicz family)**.** Let $p = (p_k)_{k\geq 0}$ satisfy $0 < H_1 \leq p_k \leq H_2 < \infty$. Define $\psi_k(u) := \psi(u^{p_k})$ and $\Psi = (\psi_k)_k \geq_0$. Set

$\ell_\psi(p) := \ell(\Psi)$,
equipped with the Luxemburg functional q_Ψ from (4). Equivalently,

$$\ell_\psi(p) = \left\{x \in \omega : \exists \mu > 0 \text{ s.t. } \sum_{k=0}^{\infty} \psi((\mu|x_k|)^{p_k}) < \infty\right\}.$$

Proposition 4.4 (Inheritance of uniform Δ_2)**.** If ψ satisfies $\psi(2u) \le K\psi(u)$ for all $u \ge 0$ and $0 < H_1 \le p_k \le H_2 < \infty$, then $\Psi = (\psi_k)$ satisfies a uniform Δ_2 condition (5).
Proof. For $u \ge 0$,

$$\psi_k(2u) = \psi\ \ (2u)^{p_k}\big) = \psi\ \ 2^{p_k}u^{p_k}\big) \le \psi\ \ 2^{H_2}u^{p_k}\big).$$

Iterating the Δ_2 inequality finitely many times yields $\psi(2^{H2}s) \le K'\psi(s)$ with K' depending only on K,H_2. Hence $\psi_k(2u) \le K'\psi_k(u)$ uniformly in k.

Corollary 4.5 (BK-property of $\ell_\psi(p)$)**.** Under the hypotheses of Proposition 4.4, $(\ell_\psi(p),q_\Psi)$ is a BK-space.
Proof. Combine Proposition 4.4 with Theorem 3.1.

5. Orlicz-paranormed matrix domains

Definition and completeness

Let $A = (a_{nk})_{n,k \ge 0}$ be lower triangular with $a_{nn} \ne 0$.

Definition 5.1 (Matrix domain of ℓ_ψ)**.** Assume ψ satisfies a Δ_2 condition. Define

$\ell_\psi(A) := \{x \in \omega : Ax \in \ell_\psi\}, q_\psi^A(x) := q_\psi(Ax).$

Theorem 5.2 (Completeness and BK-property)**.** If (ℓ_ψ,q_ψ) is a BK-space and A is lower triangular with nonzero diagonal, then $(\ell_\psi(A),q_\psi^A)$ is a BK-space; in particular it is complete.
Proof. The map T_A: $\ell_\psi(A) \to \ell_\psi$, $T_A(x) = Ax$, is linear and injective. Since A has nonzero diagonal and is triangular, A^{-1} exists as a triangular matrix with finite rows, and T_A is a linear isomorphism onto its range. By definition $q_\psi^A(x) = q_\psi(T_Ax)$, so T_A is an isometric embedding.
Completeness and continuity of coordinates transfer from ℓ_ψ to $\ell_\psi(A)$ via T_A and A^{-1}.

Inclusion principles: dominance and weights

Definition 5.3 (Global dominance)**.** For Orlicz functions ψ_1,ψ_2, write $\psi_2 \preceq \psi_1$ if there exist $a > 0$ and $C \ge 1$ such that
$\psi_2(u) \le C\ \psi_1(au)$ $(u \ge 0)$.

Proposition 5.4 (Inclusion under dominance)**.** Let ψ_1,ψ_2 satisfy Δ_2 and assume $\psi_2 \preceq \psi_1$. Then $\ell_{\psi1} \subseteq \ell_{\psi2}$ continuously, and for any triangular A with nonzero diagonal, $\ell_{\psi1}(A) \subseteq \ell_{\psi2}(A)$ continuously.

Proof. Let $y \in \ell_{\psi1}$ so $^{P}{}_{k}\ \psi_1(\lambda|y_k|) < \infty$ for some λ. Dominance gives $\psi_2((\lambda/a)|y_k|) \le C\psi_1(\lambda|y_k|)$; summing yields $y \in \ell_{\psi2}$. Continuity follows by comparing Luxemburg functionals using dominance and Δ_2 (standard; see [2]). The matrix-domain inclusion follows by applying the first part to Ax.

Definition 5.5 (Weighted Orlicz sequence space)**.** For weights $w = (w_k)_{k\ge0}$ with $w_k > 0$, define

$$\ell_\psi(w) := \left\{x \in \omega : \sum_{k=0}^{\infty} \psi(w_k|x_k|) < \infty\right\}.$$

Lemma 5.6 (Monotonicity in weights)**.** If $0 < w_k \le v_k$ for all k, then $\ell_\psi(v) \subseteq \ell_\psi(w)$ continuously.
Proof. Since ψ is nondecreasing, $\psi(w_k|x_k|) \le \psi(v_k|x_k|)$ termwise, so the modular for w is bounded by the modular for v. Luxemburg functional comparison is immediate.

6. Duals and Schauder bases in Orlicz-paranormed matrix domains

Kothe-Toeplitz duals

For a sequence space $X \subseteq \omega$ define:

$X^{\alpha} := \{y \in \omega : \sum_k |x_ky_k| < \infty \forall x \in X\}$,

$X^{\beta} := \{y \in \omega : \sum_k x_ky_k$ converges for all $x \in X\}$, $X^{\gamma} := \{y \in \omega : \sup_n |\sum_{k=0} x_ky_k| < \infty \forall x \in X\}$.

Proposition 6.1 (Duals of ℓ_ψ)**.** If ψ satisfies Δ_2, then

$(\ell_\psi)^\alpha = (\ell_\psi)^\beta = (\ell_\psi)^\gamma = \ell_{\psi^*}$.

Proof sketch. Young's inequality (1) yields $\ell_\psi * \subseteq (\ell_\psi)^\alpha$. Conversely, the classical representation of the K¨othe dual of an Orlicz sequence space identifies continuous coordinatewise functionals with $\ell_\psi *$ under Δ_2; see [2, 4, 5].

Theorem 6.2 (Duals of $\ell_\psi(A)$)**.** Let ψ satisfy Δ_2 and let A be triangular with nonzero diagonal.
Let $B = A^{-1} = (b_{nk})$. Then

$(\ell_\psi(A))^\alpha = (\ell_\psi(A))^\beta = (\ell_\psi(A))^\gamma = \{y \in \omega: yB \in \ell_\psi *\}$,

where $(yB)_n := \sum_{k=n}^{\infty} y_k b_{kn}$.

Proof. For matrix domains of BK-spaces, Kothe-Toeplitz duals transform by the inverse matrix:
$X_A^\star = \{y : \ yB \in X^\star\}$ for $\star \in \{\alpha\beta \ \ ,\gamma\}$. Applying this with $X = \ell_\psi$ and using Proposition 6.1 gives the result.

Schauder bases

Proposition 6.3 (Canonical basis of ℓ_ψ)**.** If ψ satisfies Δ_2, then the unit vectors $e^{(n)} = (\delta_{kn})_{k\geq 0}$ form a Schauder basis of ℓ_ψ.
Proof sketch. Since c_{00} is dense in ℓ_ψ and coordinate functionals are continuous (Corollary 4.2), the standard coordinate projections $S_N(x) = \sum_{n=0}^{N} x_n e^{(n)}$ converge to x in the Luxemburg topology; see [2].

Theorem 6.4 (Schauder basis of $\ell_\psi(A)$)**.** Let ψ satisfy Δ_2 and A be triangular with nonzero diagonal. Let $B = A^{-1} = (b_{nk})$ and define $e_A^{(n)} := (b_{kn})_{k\geq 0}$ *(the* n-th column of B). Then $(e_A^{(n)})_{n\geq 0}$ is a Schauder basis of $\ell_\psi(A)$, and every $x \in \ell_\psi(A)$ has the unique expansion

$$x = \sum_{n=0}^{\infty} (Ax)_n e_A^{(n)} \qquad \text{with convergence in } (\ell_\psi(A), q_\psi^A).$$

Proof. The map $T_A: \ell_\psi(A) \to \ell_\psi$, $T_A(x) = Ax$, is an isometric isomorphism onto ℓ_ψ. Transport the canonical basis of ℓ_ψ via $T_A^{-1} = B$: $e_A^{(n)} = Be^{(n)}$ is exactly the n-th column of B. Coefficients are forced to be $(Ax)_n$ by applying A and using uniqueness in ℓ_ψ.

7. Concluding remark

The preceding results provide a modular toolkit that simultaneously handles: (i) non-power growth (Orlicz control), (ii) coordinatewise variability (Maddox-type exponents), and (iii) matrixgenerated domains relevant in summability and operator theory. In concrete applications (e.g. N¨orlund matrices attached to special integer sequences), Theorem 6.2 and Theorem 6.4 reduce dual and basis computations to explicit formulas for A^{-1}.

References

1. Woo J. On modular sequence spaces. Studia Math. 1973;58:271-289.
2. Musielak J. Orlicz spaces and modular spaces. Lecture Notes in Mathematics, Vol. 1034. Berlin: Springer; 1983.
3. Parashar SD, Choudhary B. Sequence spaces defined by Orlicz functions. Indian J Pure Appl Math. 1994;25(4):419-428.
4. Lindenstrauss J, Tzafriri L. Classical Banach spaces I: sequence spaces. Berlin: Springer; 1977.
5. Rao MM, Ren ZD. Theory of Orlicz spaces. New York: Marcel Dekker; 1991.