International Journal of Statistics and Applied Mathematics 2025; 10(7): 58-67

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2025; 10(7): 58-67

https://www.mathsjournal.com
Received: 27-05-2025
Accepted: 30-06-2025

FN Mburu
School of Pure and Applied Sciences, Kirinyaga University, Kerugoya, Kenya

PW Njori
School of Pure and Applied Sciences, Kirinyaga University, Kerugoya, Kenya

CN Gitonga
School of Pure and Applied Sciences, Kirinyaga University, Kerugoya, Kenya

Corresponding Author:
FN Mburu
School of Pure and Applied Sciences, Kirinyaga University, Kerugoya, Kenya

Study of W_3 curvature tensor on Lorentzian Para Kenmotsu manifolds

FN Mburu, PW Njori, and CN Gitonga

DOI: https://www.doi.org/10.22271/maths.2025.v10.i7a.2100

Abstract
The paper investigates W_3 Curvature properties of Lorentzian Para Kenmotsu manifolds satisfying the following conditions, $W_3 - flatness$, $\xi{-}W_3 = 0$, W_3 −symmetric, W_3—semi symmetric. The behaviors of Ricci operators on Lorentzian Para Kenmotsu manifolds under the condition $W_3.Q = 0$ and their results were discussed. Expressions for these curvature tensors while considering the condition $W_3(\xi, X)$ are derived. $W3$ −flatness and connections to Einstein, η −Einstein, and special η- Einstein are investigated. These findings enhance our understanding of the geometric properties of Lorentzian Para Kenmotsu manifold in W_3 − curvature tensors.

Keywords: Para-contact metric manifold, Lorentzian almost Paracontact manifold, Lorentzian Para-Kenmotsu manifold, Einstein manifold, η −Einstein manifold, W_3-*curvature tensor*, $W_3 - flat, \xi - W_3$ *flat*, $\phi.W_3 - flat$, *and* W_3 *-semi-symmetric*

1. Introduction

In the mathematical field of differential geometry, a Kenmotsu manifold is almost contact manifold endowed with a certain kind of Riemannian metric. Notation of an almost para-contact manifold was first discussed by Sato I (2021). Many geometers in recent years have published concepts on Paracontact metric manifolds. Para Kenmotsu and special Para Kenmotsu manifolds also referred to as almost para contact metric manifolds were discussed by Singh and Sai Prasad in 1989. They pronounced significant characterizations of Para Kenmotsu manifolds. Significant properties of para Kenmotsu manifolds have attracted many geometers. Lorentzian Para Kenmotsu manifolds known as Lorentzian almost para-contact metric manifolds were introduced in 2018 and the concept of Quasi-concircular curvature tensor on Lorentzian Para Kenmotsu manifolds was considered.

Mburu F. N., *et al.* studied W_9 curvature tensor on LP Kenmotsu manifold and satisfied the following conditions; $Q.W_9 = 0$, $W_9.Q = 0$, $R.W_9 = 0$, *and* $W_9.W_9 = 0$ (2024). Rajesh Chaudhary and P.N. Singh discussed W_8- Curvature Tensor on LP Kenmotsu manifolds and discussed these results $W_8.Q = 0, \phi - W_8$ $semisymetric$ and $W_8 - flatness$. (2024).

In 1973 Pokhariyal introduced the notion of a new curvature tensor, denoted by W_3 and studied its relativistic significance and defined as

$$W_3(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(Y,Z)QX - S(X,Z)Y] \qquad (1.0)$$

Where Q is the symmetric endomorphism of the tangent space at every point, S is the Ricci tensor of type (0, 2) and X, Y and Z are vector fields on M.

Where:

$$QX = (n-1)X \qquad (1.1)$$

Which plays an important role in the theory of the projective transformations of connections.

2. Preliminaries

An n −dimensional differentiable manifold M admitting a (ϕ, ξ, η, g), $(1, 1)$ tensor field ϕ, contravariant vector field ξ, a 1 −form η and the Lorentzian metric g is called Lorentzian almost Paracontact manifold [7] if it satisfies

$$\phi^2 X = X + \eta(X)\xi \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$\phi\xi = 0 \tag{2.3}$$

$$rank\phi = n - 1 \tag{2.4}$$

$$g(X, \xi) = \eta(X) \tag{2.5}$$

Additionally, in the Lorentzian almost Paracontact manifold we have,

$$\Phi(X, Y) = \Phi(Y, X) \tag{2.6}$$

Where the fundamental 2-form Φ is defined by

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.7}$$

From (2.6) we have

$$g(X, \phi Y) = g(\phi X, Y) \tag{2.8}$$

$$g(X, \phi^2 Y) = g(\phi X, \phi Y) \tag{2.9}$$

And from (2.9) we have

$$g(\phi X, \phi Y) = g(X, Y) + \eta(Y)\eta(X) \tag{2.10}$$

A Lorentzian almost Paracontact manifold M is called Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifold [3] if

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.11}$$

For any vector fields X, and Y on M and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g. Furthermore, on the LP-Kenmotsu manifold, the following relations hold [4]

$$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi \tag{2.13}$$

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y) \tag{2.14}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \tag{2.15}$$

$$R(\xi, X)\xi = X + \eta(X)\xi = -\nabla_X \xi, \tag{2.16}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(Y)X, \tag{2.17}$$

$$S(X, \xi) = (n - 1)\eta(X), \tag{2.18}$$

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.19}$$

$$Q\xi = (n - 1)\xi \tag{2.20}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \tag{2.21}$$

For any vector fields X, Y, Z on M and where S is Ricci Tensor and Q, Ricci operator and R Curvature tensor with respect to Levi-Civita connection ∇.

A Lorentzian Para-Kenmotsu manifold M is said to be an $\eta -$ Einstein manifold if its Ricci-tensor $S(X, Y)$ is of the form (2021) [7]

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{2.22}$$

Where a and b are scalar functions on M. In particular, if $b = 0$, then the manifold is said to be an Einstein manifold.

3. A W_3-flat L.P Kenmotsu manifold

Definition 3.1 An n-dimensional L. P. Kenmotsu manifold is said to be $W_3 -$ flat if its W_3 $-$curvature tensor satisfies the following condition

$$W_3(X,Y)Z = 0$$

Theorem 3.1; W_3 -flat L.P -Kenmotsu manifold is an η $-$Einstein manifold.

Proof

Suppose the LP-Kenmotsu manifold is W_3 $-$flat, then the following hold

$$W_3(X,Y)Z = 0$$

$$\therefore W_3\ (X,Y)Z = R(X,Y)Z + \frac{1}{n-1}(g(Y,Z)QX - S(X,Z)Y) = 0 \quad (3.0)$$

$$(n-1)R(X,Y)Z = S(X,Z)Y - (n-1)(g(Y,Z)X$$

$$S(X,Z)Y = 2(n-1)g(Y,Z)X - (n-1)g(X,Z)Y \quad (3.1)$$

Putting $Y = \xi$ (3.1) in equation (3.1) gives:

$$S(X,Z)\xi = 2(n-1)\eta(Z)X - (n-1)g(X,Z)\xi \quad (3.2)$$

Contraction (3.2) yields

$$S(X,Z) = -(n-1)g(X,Z) - 2(n-1)\eta(Z)\eta(X) \quad (3.3)$$

This is η $-$Einstein manifold.

4. $\xi - W_3$ Flat L.P Kenmotsu manifold

Definition 4.1 An n-dimensional LP-Kenmotsu manifold is said to be $\xi - W_3$-flat if its W_3-curvature tensor satisfies the following condition [5].

Theorem: 4.1 A $\xi - W_3$-flat LP-Kenmotsu manifold is a special type of $\eta -$ Einstein manifold.
Proof:

$$W_3(X,Y)\xi = 0$$

Suppose W_3 (X, Y)ξ=0

$$W_3\ (X,Y)\xi = R\ (X,Y)\xi + \frac{1}{n-1}\ [g(Y,\xi)QX - S(X,\xi)Y] = 0 \quad (4.1)$$

Therefore;

$$R\ (X,Y)\xi = \frac{1}{n-1}[S(X,\xi)Y - g(Y,\xi)QX] \quad (4.3)$$

$$\eta(Y)X - \eta(X)Y = \frac{1}{n-1}[S(X,\xi)Y - \mathrm{g}(Y,\xi)QX] \quad (4.4)$$

$$\eta(Y)X - \eta(X)Y = \frac{1}{n-1}[(n-1)g(X,\xi)Y - (n-1)\eta(Y)X] \quad (4.5)$$

$$\{\eta(Y)X - \eta(X)Y\} = \eta(X)Y - \eta(Y)X \quad (4.6)$$

$$\eta(Y)X = \eta(X)Y$$

$$\eta(Y)g(X,U) = \eta(X)g(Y,U) \quad (4.7)$$

Putting $X = \xi$ yields

$$g(Y,U) = -\eta(Y)\eta(U) \quad (4.8)$$

$$S(Y,U) = -(n-1)\eta(Y)\eta(U) \quad (4.9)$$

Equation (4.9) is a Special η $-$Einstein
Thus, the theorem

5. $W_3.Q$ Lorentzian Para Kenmotsu manifold

Definition 5.1 A W_3 −L.P Kenmotsu manifold is such that

$$W_3.Q = 0$$

Theorem 5.1 An n-dimensional Lorentzian Para Kenmotsu manifold is such $W_3.Q = 0$

Proof

Suppose

$$W_3(X,Y).QZ = W_3(X,Y)QZ - Q(W_3(X,Y)Z$$

Hence

$$W_3(X,Y)QZ - Q(W_3(X,Y)Z = \mathrm{R}(X,Y)QZ - \mathrm{Q}(\mathrm{R}(X,Y)Z + \frac{1}{n-1}[g(Y,QZ)QX - S(X,Z)QY] - \frac{1}{n-1}[g(Y,QZ)QX - \mathrm{Q}(\mathrm{S}(X,Z)Y] \quad (5.1)$$

$$= \mathrm{R}(X,Y)QZ - Q(R(X,Y)Z + \frac{1}{n-1}[(g(Y,QZ)QX) - [g(Y,QZ)QX] \quad (5.2)$$

$$= R(X,Y)QZ - Q(R(X,Y)Z \quad (5.3)$$

From

$$R(X,Y)QZ = g(Y,QZ)X - g(X,QZ)Y \quad (5.4)$$

And $Q(R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY$ (5.5)

We have

$$W_3(X,Y)QZ = 0 \quad (5.6)$$

Hence $W_3 \cdot Q = 0$

6. W_3 - Symmetric Lorentzian Para Kenmotsu Manifold

Definition 6.1 A Lorentzian Para Kenmotsu manifold is called semi-symmetric if,
$(R(X,Y)R) \cdot R)(U,V)Z = 0.$

Definition 6.2 A Lorentzian Para Kenmotsu manifold is called W_3 -semi-symmetric if
$(R(X,Y)R) \cdot W_3)(U,V)Z = 0.$

Theorem 6.1 W_3 − semi-symmetric Lorentzian Para Kemontsu manifold
is a special η −Einstein manifold

Proof

Given that

$R(X,Y) \cdot W_3(U,V)Z$=0 (6.1)

Then,

$R(X,Y)W_3(U,V)Z - W_3(R(X,Y)U,V)Z - W_3(U,R(X,Y)V)Z - W_3(U,V)R(X,Y)Z = 0$ (6.2)

Putting $U = \xi$ in (6.2) we get

$$R(X,Y)W_3(\xi,V)Z - W_3(R(X,Y)\xi,V)Z - W_3(\xi,R(X,Y)V)Z - W_3(\xi,V)R(X,Y)Z = 0 \quad (6.3)$$

Since,

$$W_3(\xi,V)Z = 2g(V,Z)\xi - 2\eta(Z)V \quad (6.4)$$

we simplify each term separately to get:

First term: $R(X,Y)W_3(\xi,V)Z$

$R(X,Y)W_3(\xi,V)Z = g(Y,W_3(\xi,V)Z)X - g(X,W_3(\xi,V)Z)Y$

$= g(Y,[2g(V,Z)\xi - 2\eta(Z)V])X - g(X,[2g(V,Z)\xi - 2\eta(Z)V])Y$

$= 2\eta(Y)g(V,Z)X - 2\eta(Z)g(Y,V)X - 2\eta(X)g(V,Z)Y + 2\eta(Z)g(X,V)Y$

$= [2g(Y,\xi)g(V,Z)X - 2\eta(Z)g(Y,V)X] - [2g(X,\xi)g(V,Z)Y - 2\eta(Z)g(X,V)Y$

$= [2\eta(Y)g(V,Z)X - 2\eta(Z)g(Y,V)X] - [2\eta(X)g(V,Z)Y - 2\eta(Z)g(X,V)Y]$ (6.5)

Second Term: $W_3\ (R(X,Y)\xi,V)Z$
$W_3\ (R(X,Y)\xi,V)Z = W_3([\eta(Y)X - \eta(X)Y],V)Z$
$W_3\ (R(X,Y)\xi,V)Z = R(t,V)Z + \frac{1}{n-1}[g(V,Z)Qt - S(t,Z)Y],$
$where\ t = [\eta(Y)X - \eta(X)Y]$
$W_3\ (R(X,Y)\xi,V)Z = g(V,Z)t - g(t,Z)V + [g(V,Z)t - g(t,Z)Y],$
$= 2g(V,Z)t - 2g(t,Z)V$
$\therefore W_3\ (R(X,Y)\xi,V)Z = 2g(V,Z)[\eta(Y)X - \eta(X)Y] - 2g([\eta(Y)X - \eta(X)Y],Z)V$
$= 2\eta(Y)g(V,Z)X - 2\eta(X)g(V,Z)Y - 2\eta(Y)g(X,Z)V + 2\eta(X)g(Y,Z)V$
$== 2[\eta(Y)g(V,Z)X - \eta(X)g(V,Z)Y] - 2[\eta(Y)g(X,Z)V - \eta(X)g(Y,Z)V]$
$= 2[\eta(Y)g(V,Z)X - \eta(X)g(V,Z)Y] - 2[\eta(Y)g(X,Z)V - \eta(X)g(Y,Z)V]$ (6.6)

Third Term: $W_3(\xi,R(X,Y)V)Z$
$W_3(\xi,R(X,Y)V)Z = W_3(\xi,[g(Y,V)X - g(X,V)Y])Z$
$W_3(\xi,R(X,Y)V)Z = R(\xi,t)Z + \frac{1}{n-1}[g(t,Z)Q\xi - S(\xi,Z)t]$
$where\ t = [g(Y,V)X - g(X,V)Y]$
$W_3(\xi,R(X,Y)V)Z = g(t,Z)\xi - g(\xi,Z)t + \frac{1}{n-1}[g(t,Z)Q\xi - S(\xi,Z)t]$
$= g(t,Z)\xi - \eta(Z)t + [g(t,Z)\xi - g(\xi,Z)t]$
$\therefore W_3(\xi,R(X,Y)V)Z = 2g(t,Z)\xi - 2\eta(Z)t$

Hence,
$W_3(\xi,R(X,Y)V)Z = 2g([g(Y,V)X - g(X,V)Y],Z)\xi - 2\eta(Z)[g(Y,V)X - g(X,V)Y]$
$= 2g(X,Z)g(Y,V)\xi - 2g(X,V)g(Y,Z)\xi - 2\eta(Z)g(Y,V)X + 2\eta(Z)g(X,V)Y$

Fourth Term: $W_3(\xi,V)R(X,Y)Z$
$W_3(\xi,V)R(X,Y)Z = W_3(\xi,V)[g(Y,Z)X - g(X,Z)Y]$
$W_3(\xi,V)t = R(\xi,V)t + \frac{1}{n-1}[g(V,t)Q\xi - S(\xi,t)V]$
$where\ t = [g(Y,V)X - g(X,V)Y]$
$\therefore W_3(\xi,V)t = 2g(V,t)\xi - 2\eta(t)V$
$Hence,$

$W_3(\xi,V)R(X,Y)Z = 2g(V,)\xi - 2\eta(t)V$
$= 2g(V,X)g(Y,Z)\xi - 2g(V,Y)g(X,Z)\xi - 2\eta(X)g(Y,Z)V + 2\eta(Y)g(X,Z)V$ (6.8)

Therefore, equation (6.3) reduces to:
$2\eta(Z)g(Y,V)X - 2\eta(Z)g(X,V)Y + 2\eta(X)g(Y,Z)V - 2\eta(Y)g(X,Z)V$
$[\eta(Y)g(X,Z)V - \eta(X)g(Y,Z)V]\ - [\eta(X)g(V,Z)Y - \eta(Z)g(X,V)Y] = 0$ (6.9)

Putting $Z = \xi$ in equation (6.9) gives
$[\eta(X)\eta(V)Y + g(X,V)Y] = 0$ (6.10)

Equation (6.10) further reduces to
$S(X,V) = (n-1)\ \eta(X)\eta(V)$ (6.11)
Equation (6.11) is a Special η −Einstein, and thus, completes the proof.
(6.20)

7 $\phi - W_3$ Lorentzian Para Kenmotsu Manifold

Definition7.1: An n-dimensional LP-Kenmotsu manifold is said to be $\phi - W_3$-flat if
W_3 -curvature tensor satisfies the following condition. $\phi.W_3 = 0$
$W_3(X,Y).\phi)Z = W_3(X,Y)\phi Z - \phi(W_3(X,Y)Z = 0$ (7.0)

Theorem 7.1 A W_3 Lorentzian Para Kenmotsu Manifold is a ϕ-W_3 flat manifold
Proof;
Consider $\phi - W_3$ LP-Kenmotsu manifold, then the following hold
$W_3(X,Y)\phi Z - \phi(W_3(X,Y)Z = [R(X,Y)\phi Z + \frac{1}{n-1}\{g(Y,\phi Z)\phi X - S(X,Z)\phi Y\}] - [\phi R(X,Y)Z + \frac{1}{n-1}\{g(Y,\phi Z)\phi X - S(X,Z)\phi Y\}] = 0$ (7.1)

From (3.8) we have

$R(X,Y)\phi Z + g(Y,Z)\phi X - g(X,Z)\phi Y - [-g(X,Z)\phi Y + g(Y,\phi Z)X + g(Y,Z)\phi X - g(X,Z)\phi Y = 0$ (7.2)

$= R(X,Y)\phi Z - g(Y,Z)\phi Y + g(X,Z)\phi Y$ *(7.3)*

$R(X,Y)\phi Z = g(Y,Z)\phi Y - g(X,Z)\phi Y$ (7.4)

Thus

$g(Y,Z)\phi Y - g(X,Z)\phi Y + g(Y,Z)\phi X - g(X,Z)\phi Y - [-g(X,Z)\phi Y + g(Y,\phi Z)X + g(Y,Z)\phi X - g(X,Z)\phi Y = 0$ (7.5)

$W_3(X,Y).\phi)Z = 0$

Hence proved;

8 A W_3 −Lorentzian Para Kenmotsu manifold satisfying the condition $W_3 . R = 0$

Definition 8.1 A W_3 −Lorentzian-para-Kenmotsu manifold is said to satisfy $W_3. R = 0$ condition if

$(W_3(U,V) \cdot R)(X,Y)Z = 0$ (8.1)

Theorem 8.1, A W_3 −Lorentzian-para-Kenmotsu manifold satisfying the condition $W_3. R = 0$ is an Einstein manifold.

Proof;

The above equation (8.1) can be written as follows:

$W_3(U,V)R(X,Y)Z - R(W_3(U,V)X,Y)Z - R(X,W_3(U,V)Y)Z$

$-R(X,Y)W_3(U,V)Z = 0$ (8.2)

Putting $U = \xi$ in (8.2) we get

$W_3(\xi,V)R(X,Y)Z - R(W_3(\xi,V)X,Y)Z - R(X,W_3(\xi,V)Y)Z - R(X,Y)W_3(\xi,V)Z = 0$ (8.3)

But;

$W_3(\xi,V)W = g(V,W)\xi - g(\xi,W)V + g(V,W)\xi - g(\xi,W)V$

$W_3(\xi,V)W = 2g(V,W)\xi - 2\eta(W)V$ (8.4)

Computing the four terms in (8.3) separately gives

First term: $W_3(\xi,V)R(X,Y)Z$

$W_3(\xi,V)W = 2g(V,W)\xi - 2\eta(W)V = W_3(\xi,V)R(X,Y)Z$

$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y$ (8.5)

Using (8.4) and (8.5)

$W_3(\xi,V)R(X,Y)Z = 2g(V,g(Y,Z)X - g(X,Z)Y)\xi - [2\eta(g(Y,Z)X - g(X,Z)Y)V]$

$= 2g(V,X)g(Y,Z)\xi - 2g(V,Y)g(X,Z))\xi - 2\eta(X)(g(Y,Z)V + 2\eta(Y)g(X,Z)V$ (8.6)

Second Term: $R(W_3(\xi,V)X,Y)Z$

Using (8.3) we get

$R(W_3(\xi,V)X,Y)Z = R(W,X,Y)Z$

$R(W,Y)Z = g(Y,Z)W - g(W,Z)Y$

$W_3(\xi,V)X = 2g(V,X)\xi - 2\eta(X)V$

$R(W_3(\xi,V)X,Y)Z = g(Y,Z)2g(V,X)\xi - 2\eta(X)V - [g(2g(V,X)\xi - 2\eta(X)V, Z)Y]$

$= g(Y,Z)2g(V,X)\xi - 2\eta(X)Vg(Y,Z) - g(Z,Y)(2g(V,X)\xi + g(Z,Y)(2\eta(X)V = 0$ (8.7)

Third Term: $R(X,W_3(\xi,V)Y)Z$

$R(X,W)Z = g(W,Z)X - g(X,Z)W$

$W_3(\xi,V)Y = 2g(V,Y)\xi - 2g(\xi,Y)V$

$R(X,W_3(\xi,V)Y)Z = 2g(Z,X)(g(V,Y)\xi - 2g(Z,X)g(\xi,Y)V - 2g(X,Z)g(V,Y)\xi + 2g(X,Z)g(\xi,Y)V = 0$ (8.8)

Fourth Term: $R(X,Y)W_3(\xi,V)Z$

$W_3(\xi,V)Z = 2g(V,Z)\xi - 2\eta(Z)V$

$R(X,Y)W_3(\xi,V)Z = g(Y,[2g(V,Z)\xi - 2\eta(Z)V])X - g(X,[2g(V,Z)\xi - 2\eta(Z)V])Y$ (8.9)

$= [2g(Y,\xi)g(V,Z)X - 2\eta(Z)g(Y,V)X] - [2g(X,\xi)g(V,Z)Y - 2\eta(Z)g(X,V)Y$

$= [2\eta(Y)g(V,Z)X - 2\eta(Z)g(Y,V)X] - [2\eta(X)g(V,Z)Y - 2\eta(Z)g(X,V)Y]$ (8.10)

Simplifying (8.6), (8.7), (8.8) and (8.10) together reduces equation (8.3) to

$2g(V,X)g(Y,Z)\xi - 2g(V,Y)g(X,Z))\xi - 2\eta(X)(g(Y,Z)V + 2\eta(Y)g(X,Z)V - [2\eta(Y)g(V,Z)X - 2\eta(Z)g(Y,V)X] -$
$[2\eta(X)g(V,Z)Y - 2\eta(Z)g(X,V)Y]=0$ (8.11)

Putting $Z = \xi$ in equation (8.11) and taking inner product with ξ yields

$2g(V,X)g(Y,\xi)\xi - 2g(V,Y)g(X,\xi))\xi - 2\eta(X)(g(Y,\xi)V + 2\eta(Y)g(X,\xi)V - [2\eta(Y)g(V,\xi)X - 2\eta(\xi)g(Y,V)X] -$
$[2\eta(X)g(V,\xi)Y - 2\eta(\xi)g(X,V)Y]=0$ (8.12)

$2g(V,X)\eta(Y)\xi - 2g(V,Y)\eta(X))\xi - 2\eta(X)(\eta(Y)V + 2\eta(Y)\eta(X)V - [2\eta(Y)\eta(V)X - 2\eta(\xi)g(Y,V)X] - [2\eta(X)\eta(V)Y - 2\eta(\xi)g(X,V)Y]=0$

$2g(V,X)\eta(Y)\xi - 2g(V,Y)\eta(X))\xi - 2\eta(X)(\eta(Y)V + 2\eta(Y)\eta(X)V - [2\eta(Y)\eta(V)X + 2g(Y,V)X] - [2\eta(X)\eta(V)Y + 2g(X,V)Y = 0$ (8.13)

Putting $Y = \xi$ in equation (8.13)

$2g(V,X)\eta(\xi)\xi - 2g(V,\xi)\eta(X))\xi - [2\eta(\xi)\eta(V)X + 2g(\xi,V)X] - [2\eta(X)\eta(V)\xi + 2g(X,V)\xi=0$

$-2g(V,X)\xi - 2\eta(V)\eta(X))\xi - [-2\eta(V)X + 2\eta(V)X] - [2\eta(X)\eta(V)\xi + 2g(X,V)\xi=0$ (8.14)

$$2g(V,X)\xi = 2g(X,V)\xi$$

$S(X,V) = (n-1)g(V,X)$ (8.15)

Hence an Einstein manifold.

9 $W_3.Q$ - Lorentzian Para-Kenmotsu manifold.

Definition 9.1 A W_3 −Lorentzian Para-Kenmotsu manifold is such that $W_3.Q = 0$

Theorem 9.1. An n-dimensional LP-Kenmotsu manifold is such that $W_3.Q = 0$

Proof;

Consider that $W_3.Q = 0$

$\therefore (W(X,Y).Q)Z = W_3(X,Y)QZ - Q(W_3(X,Y)Z$

$= R(X,Y)QZ - Q(R(X,Y)Z) + \frac{1}{n-1}[Qg(Y,Z)QX - S((X,Z)QY))]$

$-\frac{1}{n-1}[Qg(Y,Z)QX - S((X,Z)QY)]$ (9.1)

$= R(X,Y)QZ - Q(R(X,Y)Z) + \frac{1}{n-1}[Qg(Y,Z)QX - S((X,Z)QY) - Qg(Y,Z)QX + S((X,Z)QY)]$ (9.2)

$= [g(Y,QZ)X - g(X,QZ)Y] - [g(Y,Z)QX - g(X,Z)QY]$

$=(n-1)[[g(Y,Z)X - g(X,Z)Y] - [g(Y,Z)X - g(X,Z)Y]]$ (9.3)

$$= 0$$

Hence proved

10. An n-dimensional Lorentzian-para-Kenmotsu manifold satisfying $Q.W_3 = 0$

Definition 10.1 An n-dimensional Lorentzian-para-Kenmotsu manifold is said to satisfy the condition $Q.W_3 = 0$ if

$(Q.W_3)(X,Y)Z = 0.$

Theorem 10.1. An n-dimensional Lorentzian Para-Kenmotsu manifold satisfying the condition $Q.W_3 = 0$ is a flat manifold.

Let us consider an LP −Kenmotsu manifold which satisfies the condition

$(Q.W_3)(X,Y)Z = 0,$

$\therefore Q(W_3(X,Y)Z) - W_3(QX,Y)Z - W_3(X,QY)Z - W_3(X,Y)QZ = 0,$ (10.1)

Computing the four terms separately gives

First term: $Q(W_3(X,Y)Z)$

$Q(W_3(X,Y)Z) = g(Y,Z)QX - g(X,Z)QY + \frac{1}{n-1}[Qg(Y,Z)QX - S(X,Z)QY]$

$= g(Y,Z)QX - g(X,Z)QY + [g(Y,Z)QX - g(X,Z)QY]$

$g(Y,Z)QX - g(X,Z)QY - g(X,Z)QY + g(Y,Z)QX$

$= 2g(Y,Z)QX - 2g(X,Z)QY$ (10.2)

Second Term: $W_3(QX,Y)Z$

$W_3(QX,Y)Z = g(Y,Z)QX - g(QX,Z)Y + \frac{1}{n-1}[Qg(Y,Z)QX - S(QX,Z)Y$

$= g(Y,Z)QX - g(QX,Z)Y + g(Y,Z)QX - g(QX,Z)Y = 2g(Y,Z)QX - 2g(QX,Z)Y$ (10.3)

Third Term: $W_3(X,QY)Z$

$W_3(X,QY)Z = g(QY,Z)X - g(X,Z)QY + \frac{1}{n-1}[g(QY,Z)QX - S(X,Z)QY$

$= g(QY,Z)X - g(X,Z)QY + g(QY,Z)X - g(X,Z)QY$

$= 2g(QY,Z)X - 2g(X,Z)QY$ (10.4)

Fourth term: $W_3(X,Y)QZ$

$W_3(X,Y)QZ = g(Y,QZ)X - g(X,QZ)Y + \frac{1}{n-1}[Qg(Y,QZ)X - S(X,QZ)Y]$

$= g(Y,QZ)X - g(X,QZ)Y + g(Y,QZ)X - g(X,QZ)Y$

$=2g(Y,QZ)X - 2g(X,QZ)Y$ (10.5)

Putting (10.2), (10.3), (10.4) and (10.5) in (10.1) yields

$$2g(Y,Z)QX-2g(X,Z)QY-[2g(Y,Z)QX-2g(QX,Z)Y]-[2g(QY,Z)X-2g(X,Z)QY]-[2g(Y,QZ)X-2g(X,QZ)Y]=0 \quad (10.6)$$

Hence proved

11. An n-dimensional $W_3 \cdot W_3$ LP-Kenmotsu manifold

Definition 11.1: Let M be an n-dimensional LP-Kenmotsu manifold. If the Riemannian curvature tensor vanishes, then M is said to be flat.

Theorem 11.1

Let M be an n-dimensional Lorentzian Para-Kenmotsu (LP-Kenmotsu) manifold. Then the curvature tensor W_3 on M satisfies the condition

$W_3 \cdot W_3 = 0$

Proof

Given,

$W_3 \cdot W_3 = W_3(U,V) \cdot W_3 = (W_3(U,V)W_3)(X,Y)Z$

But,

$$(W_3(U,V)W_3)(X,Y)Z = W_3(U,V)W_3(X,Y)Z - W_3(W_3(U,V)X,Y)Z - W_3(X,W_3(U,V)Y)Z - W_3(X,Y)W_3(U,V)Z \quad (11.1)$$

For any X,Y,Z,U,V vector field in M

Taking $U=Z=\xi$ in (11.1) we get the following relations

$$W_3(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(Y,Z)QX - S(X,Z)Y]$$

$$W_3(X,Y)\xi = R(X,Y)\xi + \frac{1}{n-1}[g(Y,\xi)QX - S(X,\xi)Y]$$

$$W_3(X,Y)\xi = \eta(Y)X - \eta(X)Y + \frac{1}{n-1}[(n-1)\eta(Y)X - (n-1)\eta(X)Y]$$

$$W_3(X,Y)\xi = 2[\eta(Y)X - \eta(X)Y]$$

Equation (11.1) reduces to

$$2W_3(\xi,V)[\eta(Y)X-\eta(X)Y] - W_3(W_3(\xi,V)X,Y)\xi - W_3(X,W_3(\xi,V)Y)\xi - W_3(X,Y)W_3(\xi,V)\xi = 0 \quad (11.2)$$

First term becomes

$$2W_3(\xi,V)[\eta(Y)X-\eta(X)Y] = 2R(\xi,V)[[\eta(Y)X-\eta(X)Y]$$
$$+\frac{2}{n-1}[g(V,[\eta(Y)X-\eta(X)Y])Q\xi + S(\xi,[\eta(Y)X-\eta(X)Y]V] \quad (11.3)$$

But,

$$2R(\xi,V)[[\eta(Y)X-\eta(X)Y] = [2\eta(Y)g(V,X)\xi - 2\eta(Y)\eta(X)V] - 2\eta(X)g(V,Y)\xi + 2\eta(X)\eta(Y)V$$
$$= [2\eta(Y)g(V,X)\xi - 2\eta(X)g(V,Y)\xi]$$

And

$$\frac{2}{n-1}[g(V,[\eta(Y)X-\eta(X)Y])Q\xi + S(\xi,[\eta(Y)X-\eta(X)Y]V] = 2[\eta(Y)g(V,X)\xi - \eta(X)g(V,Y)\xi + \eta(X)\eta(Y)V - \eta(X)\eta(Y)V]$$
$$= 2\eta(Y)g(V,X)\xi - 2\eta(X)g(V,Y)\xi$$

Hence first term (11.3) becomes

$$2W_3(\xi,V)[\eta(Y)X-\eta(X)Y] = 4\eta(Y)g(V,X)\xi - 4\eta(X)g(V,Y)\xi \quad (11.4)$$

Contracting (11.4) gives

$$\eta(2W_3(\xi,V)[\eta(Y)X-\eta(X)Y]) = -4\eta(Y)g(V,X) + 4\eta(X)g(V,Y) \quad (11.5)$$

Second term becomes

$$W_3(W_3(\xi,V)X,Y)\xi = W_3(B,Y)\xi$$

$$Where\ B = W_3(\xi,V)X = g(V,X)\xi - \eta(X)V + \frac{1}{n-1}[g(V,X)Q\xi - S(\xi,X)V]$$

$$B = 2g(V,X)\xi - 2\eta(X)V$$

But

$$W_3(B,Y)\xi = R(B,Y)\xi + \frac{1}{n-1}[g(Y,\xi)QB - S(B,\xi)Y$$

$= 2\eta(Y)B - 2\eta(B)Y$
$= 4\eta(Y)g(V,X)\xi - 4\eta(Y)\eta(X)V - 4[-g(V,X)Y - \eta(X)\eta(V)Y]$
$= 4\eta(Y)g(V,X)\xi - 4\eta(Y)\eta(X)V + [4g(V,X)Y + 4\eta(X)\eta(V)Y]$ (11.4)

Contracting (11.4) gives
$\eta(W_3(W_3(\xi,V)X,Y)\xi) = 0$

Third term becomes
$W_3(X, W_3(\xi,V)Y)\xi = W_3(X,D)\xi$
Where ,$D = W_3(\xi,V)Y$
$$W_3(\xi,V)Y = g(V,Y)\xi - \eta(Y)V + \frac{1}{n-1}[g(V,Y)Q\xi - S(\xi,Y)V]$$
$D = W_3(\xi,V)Y = 2g(V,Y)\xi - 2\eta(Y)V$

Again,
$W_3(X,D)\xi = 2\eta(D)X - 2\eta(X)D$
S$= -4g(V,Y)X - 4\eta(Y)\eta(V)X - 4\eta(X)g(V,Y)\xi + 4\eta(X)\eta(Y)V$ (11.5)

Contracting (11.5) gives
$\eta(W_3(X, W_3(\xi,V)Y)\xi) = 0$ (11.6)

Fourth term becomes
$W_3(X,Y)W_3(\xi,V)\xi = W_3(X,Y)E$

Where $E = W_3(\xi,V)\xi$

Hence,
$E = W_3(\xi,V)\xi = 2\eta(V)\xi + 2V$

Again
$$W_3(X,Y)E = R(X,Y)E + \frac{1}{n-1}[g(Y,E)QX - S(X,E)Y]$$
$= g(Y,E)X - g(X,E)Y + [g(Y,E)X - g(X,E)Y]$
$= 2g(Y,E)X - 2g(X,E)Y$
$= 4\eta(V)g(Y,\xi)X + 4g(Y,V)X - 4\eta(V)g(X,\xi)Y - 4g(X,V)Y$ (11.7)

Contracting (11.7) gives
$\eta(W_3(X,Y)W_3(\xi,V)\xi) = \eta(4g(Y,V)X - 4g(X,V)Y)$
$\eta(W_3(X,Y)W_3(\xi,V)\xi) = 4\eta(X)g\big(Y,V) - 4\eta(Y)g(X,V)\big)$ (11.8)

Substituting the results of the four terms in (11.1) above yields
$\eta(W_3 \cdot W_3) = 0$ (11.9)
$\therefore W_3 \cdot W_3 = 0$

References

1. Franco N. Lorentzian approach to noncommutative geometry [Preprint]. arXiv: 1108.0592. 2011.
2. Hedicke J. Lorentzian distance functions in contact geometry. J Topol Anal. 2024;16(2):205-225.
3. Haseeb A, Prasad R. Certain results on Lorentzian para-Kenmotsu manifolds. Bull Parana Math Soc. 2018. DOI:10.5269/bspm.40607.
4. Haseeb A, Prasad R. Certain results on Lorentzian para-Kenmotsu manifolds. Bol Soc Paran Mat. 2021;39(3):201-220.
5. Haseeb A, Ahmad M, Rizvi S. On the conformal curvature tensor of ∈-Kenmotsu manifolds. Ital J Pure Appl Math. 2018;40:656-670.
6. Devi SS, Prasad KS, Satyanarayana T. Certain curvature conditions on Lorentzian para-Kenmotsu manifolds. Reliab Theory Appl. 2022;17(2):413-421.
7. Bhatt P, Chanyal SK. LP-Kenmotsu manifold admitting Schouten-van Kampen connection.
8. DeTurck DM. Deforming metrics in the direction of their Ricci tensors. J Differ Geom. 1983;18(1):157-162.
9. Haseeb A, Prasad R. Certain results on Lorentzian para-Kenmotsu manifolds. Bol Soc Paran Mat. 2021;39(3):201-220.
10. Ali M, Khan Q, Khan AU, Vasiulla M. On generalized W3 recurrent Riemannian manifolds. Honam Math J. 2023;45(2):325-339.
11. Mburu FN, Njori PW, Gitonga CN, Moindi SK. Study of W9: curvature tensors on Lorentzian para-Kenmotsu manifolds.
12. Singh GP, Prajapati P, Mishra AK, Rajan. Generalized B-curvature tensor within the framework of Lorentzian β-Kenmotsu manifold. Int J Geom Methods Mod Phys. 2024;21(2):2450125.
13. Sarkar S, Dey S, Chen X. Certain results of conformal and *-conformal Ricci soliton on para-cosymplectic and para-Kenmotsu manifolds. Filomat. 2021;35(15):5001-5015.

14. Prasad KS, Devi SS, Deekshitulu GVSR. On a class of Lorentzian paracontact metric manifolds. Ital J Pure Appl Math. 2023;49:514-527.
15. Devi SS, Prasad KS, Satyanarayana T. Certain curvature conditions on Lorentzian para-Kenmotsu manifolds. Reliab Theory Appl. 2022;17(2):413-421.
16. Atçeken M, Karaman N. Pseudoparallel invariant submanifolds of Kenmotsu manifolds. J Amasya Univ Inst Sci Technol. 2023;4(2):72-81.