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Solution of optimal control problem by the daubechies wavelet function (Db 1)

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Abstract
The Daubechies Wavelet Function (Db 1) is also known as the Haar wavelet function. It is a sequence of "Square Shaped" functions which together form a wavelet family or basis which allows a target function over an interval to be present in terms of an orthonormal basis. In this paper, the Haar wavelets are used to solve optimal control problems by approximating the state and control variables using a basis of Haar wavelet functions. This approach transforms the original optimal control problem into a constrained nonlinear quadratic programming problem, which can then be solved using numerical methods like quasilinearization.

Keywords: Daubechies wavelet, Haar wavelet, Quasilinearization, quadratic programming

Introduction

Wavelet analysis is similar to Fourier Analysis in that it allows a target function over an interval to be represented in terms of an Orthonormal basis. The Haar sequence was proposed in 1909 by Alfred Haar. He used these functions to give an example of an orthonormal system for the space of quadrically integrable function on the interval [0, 1]. The study of wavelets, and even the term "wavelet", did not come until much later. As a special case of the family of orthogonal wavelets defining a discrete wavelet transform and characterized by a maximal number of vanishing moments for some given support, the Haar wavelet is also known as Db1. The Haar wavelet is also the simplest possible wavelet. The technical disadvantage of the Haar wavelet is that it is not continuous, and therefore not differentiable. This property can, however, be advantageous for the analysis of signals with sudden transitions (discrete signals), such as monitoring of tool failure in machines.

Overview of the Haar Wavelets

A Haar Scaling Function is a set of shifted and scaled square summable function used for defining scaling and wavelet functions in multiresolution signal analysis. It satisfies orthonormality conditions and is a solution of a specific refinement equation with two nonzero coefficients. It reconstructs the approximation of the signal at different scales. It normally captures the low frequency signal. It has a relatively smooth shape and compact support. The scaling function is defined as;

$$\phi(t) = \begin{cases} 1, & \text{if} \quad t \in [0,1] \\ 0, & \text{if} \quad t \notin [0,1] \end{cases}$$

The wavelet function is derived from the scaling function through a specific process that involves scaling and translating the scaling function. Its defined as;

$$\varphi(t)=\begin{cases}1, & \text{if } t\in[0,0.5]\\ -1 & \text{if } t\in[0.5,1]\\ 0 & \text{if } t\varnothing[0,1]\end{cases}$$

The wavelet function often has a more localized and oscillating shape than the scaling function.
For every pair $n,k \in \mathbb{Z}$ the Haar function $\Psi_{n,k}(t)$ is defined on the real line $\mathbb{R}$ by the formula;

$$\psi_{n,k}(t) = 2^{\frac{n}{2}}\,\psi(2^n t - k), t \in \mathbb{R}.$$

Which is actually supported on the right open interval, $I_{n,k}=[k2^{-n},(k+1)2^{-n}]$, that is, it vanishes whenever outside the interval. Its integral is zero and norm is one in the Hilbert space $L^2(\mathbb{R})$

$$\int_{\mathbb{R}} \psi_{n,k}(t)dt = 0, \left\|\psi_{n,k}\right\|_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \psi_{n,k}(t)^2\, dt = 1$$

The Haar functions are piecewise orthogonal, $\int_{\mathbb{R}} \psi_{n_1,k_1}(t)\psi_{n_2,k_2}(t)dt = \delta_{n_1 n_2}\delta_{k_1 k_2}$ when the two supporting intervals I_{n_1,k_1} and I_{n_2,k_2} are not equal, they are neither disjoint, or else either of the two supports, say I_{n_1,k_1}, is contained in the lower or in the upper half of the other interval, on which the function ψ_{n_2,k_2} remains constant. It follows in this case that the product of these two Haar functions is a multiple of the first Haar function, hence the product has integral zero. The Haar system on the real line is the set of functions $\{\psi_{n,k}(t): n \in \mathbb{Z}, k \in \mathbb{Z}\}$. Its complete in $L^2(\mathbb{R})$: the Haar system on the line is an orthonormal basis in $L^2(\mathbb{R})$.

The Haar Wavelet method

Let $t\in[A,B]$ where A and B are constants. Define $m=2^J$ where J is the maximal level of resolution. The interval [A,B] is distributed in $2m$ submanifolds of equal length; $\Delta t=\dfrac{(B-A)}{2m}$

The other two parameters are; the dilation parameter; $j=0,1,\ldots\ldots,J$ and the translation parameter $k=0,1,\ldots\ldots,m-1$ where $m=2^J$ and the wavelet number is given as; $i=m+k+1$

The Haar wavelet is given by;

$$h_i(t)=\begin{cases}1, & \text{for } t\in[\xi_1(i),\xi_2(i)]\\ -1, & \text{for } t\in[\xi_1(i),\xi_2(i)]\\ 0, & \text{elsewhere}\end{cases} \tag{1}$$

where $\xi_1(i)=A+2k\mu\Delta t,\quad \xi_2(i)=A+(2k+1)\mu\Delta t,\quad \xi_3(i)=A+2(k+1)\mu\Delta t$ and $\mu=\dfrac{M}{m}$, specifically, $i=1$ corresponds to the scaling faction $h_1(t)=0$ for $t\in[A,B]$. The following integral is required ;

$$P_i(t)=\int_A^t h_i(t)\,dt \tag{2}$$

These integrals can be evaluated analytically to obtain;

$$P_i(t)=\begin{cases}0 & \text{for } t\le\xi_1(i)\\ t-\xi_1(i) & \text{for } t\in[\xi_1(i),\xi_2(i)]\\ -t-\xi_1(i)+2\xi_2(i) & \text{for } t\in[\xi_1(i),\xi_3(i)]\\ 0 & \text{for } t\ge\xi_3(i)\end{cases} \tag{3}$$

With $i>1$, in case $i=1$ we have $\xi_1=A,\xi_2=\xi_3=B$ and hence ;

$$P_1(t) = t - A \quad (4)$$

So that the collocation points become;

$$t_1 = 0.5(\hat{t}_{\ell-1} + \hat{t}_\ell) \qquad \ell = 1, 2, ..., 2M \quad (5)$$

The symbol $\hat{t}_\ell$ denotes the ℓ^{th} grid point ; $\hat{t}_\ell = A + \ell\Delta t$. Eqns. (1)-(4) are discretized by replacing $t \to t_\ell$. The Haar matrices $H(i,\ell) = h_i(t_\ell)$ are then introduced so that; $P(i,\ell) = P_i(t_\ell)$ the values of P_i at $t = B$ are necessary for solution of boundary value problems. Using eqns. (3) and (4) we obtain;

$$P_i\ (B) = \begin{cases} B - A & \text{for } i = 1 \\ 0 & \text{for } i \neq 1 \end{cases}$$

Let R be the matrix ; $R(i,\ell) = P(i,\ell) - P_i\ (B)$ let also $E = \{1, 1, ..., 1\}$, $E_1 = \{1, 0, 0, ..., 0\}$ be any vectors with $t = (t\ell)$, $\hat{t} = t - AE$ So that the following holds ;

$$E_1 = \frac{E}{H}, \quad (\frac{E}{H})P = E_1 P = t - AE = \hat{t}$$

$$\frac{1}{\alpha!}(\frac{\hat{t}^\alpha}{H})P = \frac{1}{(\alpha+1)}\hat{t}^{\alpha+1}$$

Then we need to integrate by the Haar wavelet method of the equation;

$$\dot{x} = f(t, x, \mu), \quad x = (x_i) \quad (6)$$

so that we seek a solution in the form;

$$\dot{x} = aH \quad (7)$$

and by integration we obtain;

$$x = aP + c \quad (8)$$

where *c* is the vector , an integration constant. Putting eqns. (7) and (8) into (6) we obtain a system of 2*m* equations which will enable us obtain the wavelet coefficient $a = (a_i)$

Consider the problem;

$$\int_0^2 x\ dt \to \int_0^2 \ddot{x}\ dt = 1$$

with Boundary conditions; (12)

$$x(0) = \dot{x}(0) = 0, \quad \ddot{x}(2) = 0$$

where we interpret $\ddot{x}$ as control state variables; $x_1 = x_2, \quad x_2 = \dot{x}, \quad x_3 = \ddot{x} = u, \quad x_4 = \int_0^t u^2\ dt$

The state equations are given by;

$$x_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u, \quad x_4 = u^2;$$

with boundary conditions; (13)

$$x_1(0) = x(0)_2 = x_4(0) = 0, \quad x_3(2) = 0, \quad x_4(2) = 1$$

with the Hamiltonian given by; $\hat{H} = -x_1 + \Psi_1 x_2 + \Psi_2 x_3 + \Psi_3 u + \Psi_4 u^2$

The adjoint system will now take the form;

$$\dot{\Psi}_1 = E,\ \dot{\Psi}_2 = -\Psi_1,\ \dot{\Psi}_3 = -\Psi_2,\ \dot{\Psi}_4 = 0 \tag{14}$$

with the transversality conditions $\Psi_1(2) = \Psi_2(2) = \Psi_3(2) = 0$

Firstly, we integrate Eqns. (14) with the assumption that;

$$\begin{aligned} &\dot{\Psi}_1 = b_1 H, \qquad \Psi_1 = b_1 R \\ &\dot{\Psi}_2 = b_2 H \qquad \Psi_2 = b_2 R \\ &\dot{\Psi}_3 = b_3 H \qquad \Psi_3(2) = b_3 R \\ &\dot{\Psi}_4 = 0 \qquad \Psi_4 = \lambda,\ \text{a constant} \end{aligned} \tag{15}$$

Matrix R is calculated from Eqn. (6), now, substituting Eqn. (15) into Eqn. (14), we obtain;

$$b_1 H = E,\ b_1 R + b_2 H = 0,\ \ b_2 R + b_3 H = 0 \tag{16}$$

Next, we calculate the wavelet coefficients $b_1,\ b_2,\ b_3$ and the adjoint variables $\Psi_1,\ \Psi_2,\ \Psi_3$ Working out the optimal control using $\frac{\partial \hat{H}}{\partial u}$ gives us; $u = -\frac{\Psi_3}{2\Psi_4} = -\frac{\Psi_3}{2\lambda}$. We already know Ψ_3, hence, we can calculate the auxiliary variable $\hat{u} = \lambda u = -\frac{\Psi_3}{2}$

The state variables are then developed into the Haar series using Eqns. (10) and (11);

$$\begin{aligned} &\dot{x}_1 = a_1 H,\ \ x_1 = a_1 P,\ \ \dot{x}_2 = a_2 H,\ \ x_2 = a_2 P \\ &\dot{x}_3 = a_3 H,\ \ x_3 = a_3 P,\ \ \dot{x}_4 = a_4 H,\ \ x_4 = a_4 P \end{aligned} \tag{17}$$

Using these Eqns. In Eqn. (13) we obtain;

$$\begin{aligned} &a_1 H - a_2 P = 0,\ \ a_2 H - a_3 P = 0 \\ &a_3 H = \frac{\hat{u}}{\lambda},\ \ a_4 H = \frac{\hat{u}^2}{\lambda^2} \end{aligned} \tag{18}$$

The Lagrange multiplier λ is calculated from Eqn. (18) as $\hat{a}_4 = \lambda^2,\ \ a_4 = \frac{\hat{u}^2}{H}$ with the boundary condition $x_4(2) = \phi_1$ to give $x_4\big|_{t=2} = a_4 p\big|_{t=2} = 2a_4(1) = 1$. These proceeding two Equations gives;

$$\lambda = \sqrt{2\hat{a}_4(1)}$$

From which the state variables are then calculated. The exact solutions are given by;

$$x_{ex} = -\frac{1}{12\lambda}[\frac{1}{120}(t-2)^6 + \frac{4}{3}(t-2)^3 - \frac{72}{5}t + \frac{152}{15}]$$

$$u_{ex} = -\frac{1}{48\lambda}(t-2)[(t-2)^3 + 32]$$

$$\lambda_{ex} = 0.7559$$

The error estimates below are then introduced for the purpose of estimating the accuracy of the results;

$$\delta_x = \max_i |x_{ex}(t_i) - x(t_i)|$$

$$\delta_u = \max_i |u_{ex}(t_i) - u(t_i)|$$

$$\delta_\lambda = |\lambda_{ex} - \lambda|$$

The computer simulations then gives;

J	δ_x	δ_u	δ_λ
4	$6.5E-4$	$6.3E-4$	$3.8E-7$
5	$1.6E-4$	$1.6E-4$	$3.8E-7$
6	$4.0E-5$	$4.0E-5$	$3.8E-7$

These results show that a small number of points (J = 4; 32 grid points) guarantees sufficient accuracy.
Brysan and Ho (1975) solved the following problem analytically.

$$I = \frac{1}{2}\int_0^1 u\ dt \to \min, x_1 \le \ell$$

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ x_1(0) = x_1(1) = 0,\ x_2(0) = -x_2(1) = 1 \quad (18)$$

where $\ell > 0$ is a given constant

We introduce Haar wavelet method of solution to compare with the analytical solution. Consider the case where $x_1(t) = \ell$ with $t \in [t_1, t_2]$. This subinterval cannot be near the boundaries $t = 0$ or $t = 1$ because in this case the boundary conditions cannot be satisfied.

Let $x_1(t) < \ell$ for some interval $t \in [0, t_1]$ and $x_1(t) = \ell$ for $t \in [t, 0.5]$. Now the Hamiltonian takes the form $\hat{H} = -\frac{1}{2}u_2 + \Psi_1 x_1 + \Psi_2 u$. The fact that $\frac{\partial \hat{H}}{\partial u} = 0 \Rightarrow \Psi = u$. Again, the adjoint system; $\dot{\Psi} = -\frac{\partial H}{\partial x} + \mu \nabla s,\ t \in [t_1, t_2]$ and in the present case $g = x_1 - \ell,\ s = x_2,\ \nabla s = (0,1)$ which leads to;

$$\dot{\Psi}_1 = 0,\ \ \dot{\Psi}_2 = -\Psi_1 + \mu \quad (19)$$

where $\mu(t) = 0$ for $t \in [0, t_1]$. Now, consider the subinterval $t \in [t_1, 0.5)$, since $x_1 = \ell$ it follows from the state equation that $x_2 = u = 0$ and consequently $\Psi_2 = 0$. When we integrate Eqn. (19), we get $\Psi_1 = c_1,\ \ \mu(t) = c_1$ where c_1 is the constant of integration. Eqn. (18) can be written as;

$$I = \frac{1}{2}\int_0^{t_1} u^2 \, dt \to \min$$

(20)

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ x_1(0) = 0,\ x_2(0) = 1,\ \ x_1(t_1) = \ell,\ x_2(t_1) = 0$$

From Eqns. (10) and (11), we seek the wavelet solution in matrix form thus;

$$\dot{x}_1 = a_1 H \quad x_1 = a_1 P \quad \dot{x}_2 = a_2 H \qquad x_2 = a_2 P + E \qquad (i)$$

(21)

$$\dot{\Psi}_1 = b_1 H \quad \Psi_1 = b_1 P + c_1 E \quad \dot{\Psi}_2 = b_2 H \qquad \Psi_2 = b_2 P + c_2 E \qquad (ii)$$

The matrix H and P is gotten from Eqns. (1), (2), and (3) assuming that $A = 0,\ \ B = t_1$ Putting Eqn. (21) into Eqn. (20) and the adjoint system (19), we obtain;

$$a_1 H = a_2 P = E \qquad a_2 H = b_2 P + c_2 E$$

$$b_1 H = 0 \qquad\qquad b_2 H = -b_1 H - c_1 E$$

which implies that; $b_1 = 0,\quad b_2 = -\frac{c_1 E}{H} = -c_1 E_1,\quad c_2 = c_1 t_1$ and due to continuity $\Psi(t_1) = 0$ we have it that $\Psi_2 = c_1(t_1 E - t) = u$

The second equation of Eqn. (21) can be integrated to obtain;

$$a_2 H = -c_1 E_1 P + c_2 E = -c_1 t + c_2 E$$

$$x_2 = -c_1(\frac{E}{H})P + (\frac{t}{H})P + E$$

Using Eqns. (7) and (8), this result can be put in the form; $x_2 = c_1 t(t_1 - 0.5t) + 1$ Since $a_2 P = -c_1(\frac{t}{H})P + c_1 t_1(\frac{E}{H})P = -c_1\frac{t^2}{2} + c_1 t_1 t$ then $a_1 = \frac{(a_2 P + E)}{H} = -c_1\frac{t^2}{2H} + \frac{c_1 t_1 t}{H} = \frac{E}{H}$ and $x_1 = -c_1\frac{t^2}{2H}P + \frac{c_1 t_1 t}{H}P + (\frac{E}{H})P$ so that $x_1 = -c_1 t^2(\frac{t}{6} - \frac{1}{2}t_1) + t$

The constants c_1 and t_1 are calculated from the boundary conditions; $x_1(t_1) = \ell,\ \ x_2(t_1) = 0$ from which we get;

$$t_1 = 3\ell, \quad c_1 = -2/t_1^2$$

$$x_1 = \ell\xi(\xi^2 - 3\xi + 3), \quad \xi = \frac{t}{t_1}$$

$$x_1 = (1-\xi)^2, \ \ u = -\frac{2}{3\ell}(1-\xi)$$

Which is actually the same result as the one obtained by Bryson and Ho (1975) [10]
Finally, suppose we now consider an optimal control problem with control inequality constraints. Let;

$$I=\int_0^1 (x_1^2+x_2^2+\alpha u^2)\,dt \to \min, \quad |u|\le u_0$$

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_2+u,\ x_1(0)=0,\ x_2(0)=-1$$

be the optimal control problem with the assumption that the control is smooth, implying that the function $u(t)$ is continuous. With the Hamiltonian $\hat{H}=-(x_1^2+x_2^2+\alpha u^2)+\Psi_1 x_2+\Psi_2(-x_2+u)$ together with the adjoint system;

$$\dot{\Psi}_1=-\frac{\partial \hat{H}}{\partial x_1}=2x_1,\quad \dot{\Psi}_2=-\frac{\partial \hat{H}}{\partial x_2}=2x_2-\Psi_1+\Psi_2 \tag{23}$$

Accordingly, $\Psi_1(1)=\Psi_2(1)=0$ in the region where $|u|<u_0$ it follows that $\Psi_2=2\alpha u$ from the admissible control conditions , that is., $\frac{\partial \hat{H}}{\partial u}=0$

Assume that $u=u_o$ for $t\in[0,t]$ and $u<u_0$ for $t\in[t,1]$ we will calculate the value of t_1 for the unknown; we shall assign some value to t_1 and subsequently integrate the state equation for $t\in[0,t_1]$

Then the matrices H and P are worked out for $(a=0,\ b=t_1)$ from Eqns. (10) and (11);

$$\begin{aligned}\dot{x}_1&=a_1H,\quad x_1=a_1P\\ \dot{x}_2&=a_2H,\quad x_2=a_2P-E\end{aligned} \tag{24}$$

Using the state equations , $\dot{x}_1=x_2$ and $\dot{x}_2=-x_2+u_0$ we obtain;

$$a_1H-a_2P=-E$$

$$a_2(H+P)=(1+u_o)E$$

Now, solving this system, we got the wavelet coefficients $a_1,\ a_2$ and calculate the function $x_1,\ x_2$ using Eqn. (24). We need the values $x_1=x_1(t)$ and $x_2=x_2(t)$ while μ follows from Eqn. (3) that $P_1(t_1)=t_1$ and $P_i(t_1)=0$ for $i\ne 1$. Using Eqn. (24), we find that $x_1^*=a_1(1)t_1$ and $x_2^*=a_2(1)t_1-1$.

Let the subinterval be divided into 2 m equal parts and we calculate the matrices H, P and R using Eqns. (11), (3) and (4) with the assumption that; $A=t_1,\ B=1-t_1$. We need a solution in the form;

$$\begin{aligned}&\dot{x}_1=\hat{a}_1H,\ x_1=\hat{a}_1P+x_1E,\ \dot{x}_2=\hat{a}_2H,\ x_2=\hat{a}_2P+x_2^*E\\ &\dot{\Psi}_1=\hat{b}_1H,\ \Psi_1=\hat{b}_1R,\ \dot{\Psi}_2=\hat{b}_2H,\ \Psi_2=\hat{b}_2R\end{aligned} \tag{25}$$

The matrix R is gotten via Eqn. (6) and $\hat{a}_1,\ \hat{a}_2,\ \hat{b}_1\ \hat{b}_2$ are the wavelet coefficients for the subinterval $t\in[t_1,1]$. Then substituting Eqn. (25) in Eqn. (22) and (23) and recalling that $\Psi_2=2\alpha u$ we obtain;

$$\begin{aligned}&\hat{a}_1H-\hat{a}_2P=x_2^*\hat{a}_2(H+P)=\frac{1}{2\alpha}\hat{b}_2R-x_2^*E\\ &-2\hat{a}_1P+\hat{b}_1H=2x_1^*E,-2\hat{a}_2P+\hat{b}_1R+\hat{b}_2(H-R)=2_2^*E\end{aligned} \tag{26}$$

Eqn. (26) is numerically solved. The fact that $u(t)$ must be continuous at $t = t_*$, however, in the case of an arbitrary chosen value, t_* the requirement is not met. This requires the estimation by the function;

$$\Delta = u(t_1 - 0) - u(t_1 + 0) \quad \because \; u(t_1 - 0) = u_0$$

$$u(t_1 + 0) = \frac{1}{2\alpha}\Psi_2(t_1 + 0) = \frac{1}{2\alpha}\hat{b}_2 R = -\frac{1}{2\alpha}\hat{b}_2(1)(1 - t_1)$$

Which then gives;

$$\Delta = \frac{1}{2\alpha}\hat{b}_2(1)(1 - t_1) + u_0 \tag{27}$$

The value of t_* was varied until the condition $\Delta = 0$ was fulfilled with the necessary exactness. Simulation was done for $u_0 = 0.5, \;\; \alpha = 0.5$. The exactness of the solution was estimated by calculating the values of $t_1, \; x_1(1), \; x_2(1)$ at different levels of resolution J as per the table below

J	t_1	$x_1(1)$	$x_2(1)$
4	0.338569	–0.48679	–0.20527
5	0.338570	–0.48676	–0.20535
6	0.338574	–0.48675	–0.20536

These are the values of parameter t_1 and boundary values of Eqn. (22). The values of compare with the simulated values hence the effectiveness of the Haar wavelet method.

Conclusion

From the three different case solutions above, it's clear that the Haar Wavelet method proves to be an effective and reliable approach for solving optimal control problems. Its piecewise constant structure enables efficient handling of complex boundary conditions and discontinuities, making it particularly well-suited for problems with non-smooth solutions. Furthermore, the method offers significant computational advantages due to its sparse matrix representation and fast convergence properties. Numerical results demonstrate that the Haar Wavelet method achieves high accuracy with relatively low computational cost, validating its applicability to a wide range of control problems in engineering and applied sciences. The method provides accurate solutions, even with a relatively small number of collocation points. This is actually in concurrence with Sengeta, Singh and Kumar (2014) [9] who also noted that the Haar wavelet method presented simple and straight forward numerical technique in solving Differential equations

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