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Some Remars on regular near-ring

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Abstract
The idea of regularity in algebraic structures was first introduced by J.C. Beidleman in 1968, followed by structural advancements made by Steve Ligh in 1970 in the context of regular near-rings. This paper explores several properties specific to regular near-rings, with an emphasis on their behavior in relation to their center, Boolean near-rings, and division near-rings. It is demonstrated that while both the center of a regular near-ring and Boolean near-rings themselves exhibit regularity, division near-rings do not conform to this property.

Keywords: Near-ring, regular near-ring, boolean near-ring, division near-ring, center of near-ring.

1. Introduction
The evolution of near-ring theory has largely been influenced by classical ring theory. Early work dates back to 1905 when Dickson examined division near-rings. Zassenhaus, in 1936, established that the additive group in any finite division near-ring is commutative, a result that was generalized by Neumann in 1940 to include all division near-rings. Despite being finite and planar, division near-rings are not always planar when infinite, as demonstrated by Zameer in 1964.
The notion of regular rings, originally introduced by Von Neumann, laid the groundwork for the concept of regularity in near-rings. This idea has since been expanded by several researchers including Beidleman, Choudhari, Goyal, Heatherly, Ligh, Mason, Murty, and Szeto, with their findings compiled in Pilz's monograph. Mason, in 1980, introduced the classifications of left, right, and strong regularity in near-rings, asserting their equivalence under certain conditions. Murty further extended the theory in 1984, while Ohori (1985) analyzed π-regularity and strong π-regularity in rings.
Research on generalized ring structures has yielded various forms of near-rings such as Boolean near-rings, IFP near-rings, left bipotent near-rings, P-strongly regular, and strong IFP near-rings. Contributions by Dickson, Zassenhaus, and Wielandt were foundational, while later classification and ideal-theoretic approaches came from Betsch, Laxton, Beidleman, Bell, Ramakotaiah, and Pilz. Ideal theory and radical structures were further refined by Groenewald, Holcombe, and Birkenmeier.
Recent advancements include the introduction of s-weakly regular and semi-central idempotent near-rings. This study focuses specifically on analyzing regular near-rings and investigating how their properties manifest in their centers, Boolean variants, and in contrast with division near-rings.
In this paper, N denotes the near-rings. Here we say that a is regular (Von-Neumann) in N if $a \in aNa$. A particular solution to $axa = a$ is called an inner inverse of a. A solution to $xax = a$ is called a outer inverse. Finally, an element that satisfies $axa = a$ and $xax = a$ is called inverse (relative) of a.

2. Key Definitions
- **Division Near-Ring**
  A near-ring N (with more than one element) is called a division near-ring if its set of nonzero elements forms a multiplicative group.

Boolean Near-Ring
A near-ring N is termed Boolean if every element a in N satisfies $a^2 = a$.

Regular Element
An element a in a near-ring is called regular if there exists x in N such that a = axa.

Center of a Near-Ring
The center of a near-ring N, denoted Z(N), consists of all elements that commute with every element in N under both addition and multiplication.

3. Main Theorems and Proof
3.1 Theorem: If N is a regular near-ring then N is not a division near-ring.
Proof: (i) A Regular near-ring is one where for every element, there exist an element y such that $x = xyx$.
(ii) A division near-ring is a near -ring with a multiplicative identity where every nonzero element has a multiplicative inverse.
Above two properties are mutually exclusive, that means no near-ring possess both. Therefore a regular near-ring N is not a division near-ring.

3.2 Theorem: If N is Boolean near-rings then N is Regular near- ring.
Proof: Let N denote Boolean near-ring. Since N is Boolean near-ring, then $a.a = a$ ∀a∈ N, so we have $aaa = aa = a$. Since every element N is a Regular element, then N is a Regular near-ring.

3.3 Theorem: If N is a division near-ring, then $(N, +)$ is abelian.
Proof: The addition operation $(+)$ within the division near-ring N is commutative i. e. For any $a, b \in N, a + b = b + a$. This is a key property of division near-ring, while they don't necessarily have a commutative multiplication; their additive structure is always abelian.

3.4. Theorem: If every element x of a regular near- ring N has multiplicative inverse, then the center of a regular near-ring is also regular.

Proof: Let $Z(N)$ be a center of regular near-ring, then $Z(N)$ is non empty set and $Z(N) \subseteq N$. To prove that $Z(N)$ is a regular near-ring, we will first prove that $(Z(N), +)$ is an abelian group.

- Suppose $x, y \in Z(N)$, Then $xr = rx$ and $yr = ry$ ∀ r ∈ N. As a consequence, we have $xr + yr = (x + y)r$ and $rx + ry = r(x + y)$. Since $xr = rx$ and $= ry$, then $xr + yr = rx + ry$. Hence $(x + y)r = r(x + y)$. It means that $x + y \in Z(N)$. So, the addition in $Z(N)$ is a Binary operation.
- Next, we need to prove that for every element $x, y, z \in Z(N)$ we also have $x + (y + z)$ and $(x + y) + z$ in $Z(N)$. Since x, $y, z \in Z(N)$, we have $xr = rx, yr = ry$ and $zr = rz$. Then,

$$xr + yr + zr = (x + y)r + zr = ((x + y) + z)r \quad \text{(i)}$$

$$rx + ry + rz = r(x + y) + rz = r((x + y) + z) \quad \text{(ii)}$$

$$xr + yr + zr = xr + (y + z)r = (x + (y + z))r \quad \text{(iii)}$$

$$rx + ry + rz = rx + r(y + z) + rz = r(x + (y + z)) \quad \text{(iv)}$$

Because of $xr = rx, yr = ry$ and $zr = rz$ then, $xr + yr + zr = rx + ry + rz$ from equation (i) and (ii), we have $\big((x + y) + z\big)r = r((x + y) + z)$ and from equation (ii) and (iv) we have $\big(x + (y + z)\big)r = r(x + (y + z))$. Then we conclude that $(x + y) + z \in Z(N)$ and $x + (y + z) \in Z(N)$.

Furthermore,
$xr + yr + zr = rx + ry + rz$, so that$\big((x + y) + z\big)r = r((x + y) + z)$, with $(x + y) + z \in Z(N)$ and $x + (y + z) \in Z(N)$.
Hence $x + (y + z)$=$(x + y) + z$.

- To show that 0 is an identity element of (N), we need to prove $0 \in Z(N)$. Firstly let, $0 = r0 = 0r, \forall r \in N$, so that $0 \in Z(N)$. Since $0 \in Z(N)$ and $Z(N) \subseteq N$, then the addition is also commutative. So we have $x + 0$ and $0 + x$, $x \in Z(N)$. Next we will prove that 0 is identity element in (N) . Suppose $x \in Z(N)$, then $xr = rx, \forall r \in N$. Then we have

$xr + 0r = (x + 0)r = (0 + x)r = xr$ and $rx + r0 = r(x + 0) = r(0 + x) = rx$.Since $xr = rx$, then $xr + 0r = rx + r0$. As a result, $(x + 0)r = r(0 + x)$. This confirm that $x + 0 = 0 + x = x \; \forall x \in Z(N)$ (4)

Suppose $x, -x \in Z(N)$ so that $rx = xr$ and $-rx = -xr$. We get

$$xr + (-xr) = rx + (-rx) = r(x - x) = r.0 = 0$$

And

$(-xr) + xr = (-rx) + rx = r\,(-x + x) = r.0 = 0$

Hence,

$xr + (-xr) = (-xr) + xr = 0.$

Likewise,

$\big(x + (-x)\big)r = \big((-x) + x\big)r = \ 0$

Which gives us

$x + (-x) = (-x) + x = 0.$

This proves that $-x$ is an inverse of $x \in Z(N)$, and this means that every $x \ \in Z(N)$ has inverse $-x \in Z(N)$.
(5) Suppose $x, y \ \in Z(N)$, so that $xr = rx$ and$yr = ry$, for every $r \in N$, we get

$xr + yr = yr + xr = (y + x)r.$

It means that $+y = y + x$, that is the addition in the Z(N) is commutative.
From (1) to (5) we conclude that $Z(N)$ is Abelian group.
Furthermore, to show that $(Z(N), +, .)$ is a Near-ring. We need to prove $(Z(N), .)$ is a semi-group. The associative property is inherited from the associative property in the Near-ring. Let $x, y \ \in Z(N)$ and $r \ \in N$. Since $Z(N) \subseteq N$, then $x, y \ \in N$. Since the multiplication is also associative in the Near-ring$N$, we get $(xy)r = x(yr) = x(ry) = (xr)y = (rx)y = r(xy)$.
Because of $(xy)r = r\,(xy)$, we get $x, y \in Z(N)$. Hence the multiplication is a binary operation in the (N) . So we get a prove that $(Z(N), .)$ is a semigroup. As Left or Right Distributive property in $Z(N)$ is given by Distributive property in Near-ring N, then we find that $(Z(N), .)$ is a Near-ring.
Now, it is remaining for us to show that every element $x \in Z(N)$is a regular element. Assume that x is an element of regular near-ring Nwhich have multiplicative inverse. If N is a regular near-ring with center $Z(N)$, then $Z(N) \subseteq N$. For every element of $x \in Z(N)$ result in x is also an element of N. Since N is a regular near-ring, then there exist $w \in N$ such that $xwx = x$. Hence we have $t = xwx$. By the property of regular near-ring:

$xtx = x(xwx)x$
$= (xwx)wx$
$= xwx$
$= x$

Since, $tr = rt$, then $t \in Z(N)$. Furthermore,

$xtx = x(xwx)x$
$= (xwx)wx$
$= xwx$
$= x$

Therefore x is an element of $Z(N)$ and $Z(N)$ is a regular near-ring.

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