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Analysis of some properties of w_5 curvature tensor in lorentzian para kenmotsu manifold

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Abstract
This paper investigates the properties of W_5 curvature tensors on Lorentzian Para Kenmotsu manifolds. Properties of this Curvature tensor under various conditions on these manifolds are explored and their geometric implications are examined. The study also includes investigations of W_5 flatness, ξ - W_5 flatness, $\phi - W_5$ flatness and W_5 -Semisymmetric on Lorentzian Para Kenmotsu manifolds and their connections to η-Einstein, Einstein and special η -Einstein. Additionally, The Ricci operator's behaviors on Lorentzian Para Kenmotsu manifolds under the conditions $W_5.Q = 0$, $Q.W_5 = 0$ are analyzed. Expressions for this curvature tensor while considering the condition $W_5(\xi, X)$. $W_5 = 0, R.W_5 = 0$ and $W_5.W_5 = 0$ are derived. Proves to determine whether these manifolds are flat will be provided. The findings of this study enhance understanding of the geometric properties of Lorentzian Para Kenmotsu manifolds in relation to W_5 curvature tensors.

Keywords: Para-contact metric manifold, Lorentzian almost Paracontact manifold, Lorentzian Para-Kenmotsu manifold, Einstein manifold, $\eta-$Einstein manifold, W_5-curvature tensor, $W_5 -$ flat, $\xi - W_5$ flat, $\phi.W_5 -$ flat, and W_5 -semi-symmetric

Introduction

In 1989, [1] K. Matsumoto introduced Lorentzian paracontact manifolds, specifically LP-Sasakian manifolds, which have since been extensively studied by various geometers. Subsequent research by Matsumoto, Mihai, Rosca, Shaikh, De, Venkatesha, Pradeep Kumar, and Bagewadi focused on these manifolds with significant results [3]. In 1995, [4] Sinha and Sai Prasad defined para-Kenmotsu and special para-Kenmotsu manifolds, akin to P-Sasakian and SP-Sasakian manifolds. Abdul Haseeb and Rajendra Prasad in 2018 introduced Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifolds, studying ϕ-semisymmetric LP-Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons [6]. Njori *et al*. did several studies on $W_8 -$ curvature tensors on various types of manifolds, including Kenmotsu manifolds [2, 8].

In 1970, [5] Pokhariyal and Mishra introduced new tensor fields on a Riemannian manifold, called the Weyl-projective curvature tensor of type (1, 3) and the tensor field E. The concept of W_5-curvature tensor was defined by Tripathi and Gupta [15] of an n (where $n = 2m + 1$)-dimensional Riemannian manifold are, respectively, defined as

$$W_5\,(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{n-1}\,[g(X, Z)S(Y, T) - g(Y, T)S(X, Z)\ (2.1)$$

For all $(X, Y, Z) \in_X (M)$. Where R represents the curvature tensor and S corresponds to the Ricci tensor of the manifold.

2. Preliminaries

An $n-$dimensional differentiable manifold M admitting a (ϕ, ξ, η, g), (1,1) tensor field ϕ, contravariant vector field ξ, a $1-$form η and the Lorentzian metric g is called Lorentzian almost Paracontact manifold [7] if it satisfies:

$\phi^2 X = X + \eta(X)\xi$ (2.1) $\eta(\xi) = -1,$ (2.2)

$\phi\xi = 0$ ` (2.3)

$rank\phi = n - 1$ (2.4)

$\phi(X, Y) = \phi(Y, X)$ (2.5)

Where

$\phi(X, Y) = g(X, \phi Y)$ (2.6)

Additionally, in the Lorentzian Para Kenmotsu manifold we have

$g(X, \phi Y) = g(\phi X, Y)$ (2.7)

$g(X, \xi) = \eta(X)$ (2.8)

$g(X, \phi^2 Y) = g(\phi X, \phi Y)$ (2.9)

From (2.1) and (2.9) we have

$g(X, \phi^2 Y) = g(X, Y + \eta(Y)\xi)$

$= g(X, Y) + \eta(Y) g(X, \xi)$

$g(\phi X, \phi Y) = g(X, Y) + \eta(Y)\eta(X)$ (2.10)

A Lorentzian almost Paracontact manifold M is called Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifold (2018) [3] if

$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X,$ (2.11)

From 2.11 replacing Y by ϕY

$(\nabla_X \phi)\phi Y = -g(\phi X, \phi Y)\xi - \eta(\phi Y)\phi X,$ (2.12)

For any vector fields X, and Y on M and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g.

Furthermore, on the LP-Kenmotsu manifold, the following relations hold: [4]

$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi$ (2.13)

$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y)$ (2.14)

$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X$ (2.15)

$R(\xi, X)\xi = X + \eta(X)\xi = -\nabla_X \xi,$ (2.16)

$R(X, Y)\xi = \eta(Y)X - \eta(Y)X,$ (2.17)

$S(X, \xi) = (n - 1)\eta(X),$ (2.18)

$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y),$ (2.19)

$Q\xi = (n - 1)\xi$ (2.20)

$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y),$ (2.21)

$S(X, Y) = g(QX, Y)$

Where Q is Ricci operator defined as $QX = (n - 1)X$

For any vector fields X, Y, Z on M and where S is Ricci Tensor and Q, Ricci operator and R Curvature tensor with respect to Levi-Civita connection ∇.

A Lorentzian Para-Kenmotsu manifold M is said to be an $\eta -$ Einstein manifold if its Ricci-tensor $S(X,Y)$ is of the form (2021) [7]

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.22}$$

where a and b are scalar functions on M. In particular, if $b = 0$, then the manifold is said to be an Einstein manifold. Additionally, special $\eta - Einstein$ manifold when $a = 0$.

3.A W_5-flat L.P Kenmotsu manifold

Definition 3.1, An n-dimensional L. P. Kenmotsu manifold M is said to be flat if it satisfies the following condition:

$$R(X,Y)Z = 0$$

Theorem 3.1, W_5 -flat L.P -Kenmotsu manifold is a flat manifold.

Proof

Suppose the LP-Kenmotsu manifold is W_5-flat, then the following hold

begin{equation}

$$\therefore W_5\,(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}(g(X,Z)S(Y,T) - S(X,Z)g(Y,T) = 0$$

$$R(X,Y,Z,T) + \frac{1}{n-1}(g(X,Z)S(Y,T) - S(X,Z)g(Y,T) = 0$$

$$R(X,Y,Z,T) + (g(X,Z)g(Y,T) - g(X,Z)g(Y,T) = 0$$

$$R(X,Y,Z,T) = 0$$

This completes the proof that a W_5-LP-Kenmotsu manifold is a flat manifold

4. A $\xi - W_5$-flat Lorentzian Para Kenmotsu manifold

Definition 4.1: An n-dimensional LP-Kenmotsu manifold is said to be $\xi - W_5$-flat if its W_5 -curvature tensor satisfies the following condition.

$$W_5(X,Y)\xi = 0$$

Theorem 4.1: An ξ -W_5 flat Lorentzian Para Kenmotsu manifold is a special type of η-Einstein manifold.
Let us consider

$$W_5(X,Y)\xi = R(X,Y)\xi + \frac{1}{n-1}\{g(X,\xi)QY - S(X,\xi)Y] = 0 \tag{4.1}$$

$$0 = \eta(Y)X - \eta(X)Y + g(X,\xi)Y - g(X,\xi)Y \tag{4.2}$$

$$\therefore \eta(X)Y = \eta(Y)X \tag{4.3}$$

$$\eta(X)g(Y,U) = \eta(Y)g(X,U) \tag{4.4}$$

Setting Y=ξ

$$\eta(X)g(\xi,U) = \eta(\xi)g(X,U) \tag{4.5}$$

$$\eta(X)\eta(U) = -g(X,U)$$

$$g(X,U) = -\eta(X)\eta(U)$$

$$S(X,U) = -(n-1)\eta(X)\eta(U) \tag{4.6}$$

Hence (4.6) is a special η-Einstein manifold

$W_3 -$ Semisymmetric Lorentzian Para Kenmotsu manifold

Definition 5.1: An n-dimensional $W_5 -$ Lorentzian-Para-Kenmotsu manifold M is is called Semisymmetric if for all vector fields X, Y, Z, U, V on M, the following holds:

$$(R(X,Y)\cdot R)(U,V)Z = 0. \tag{5.1}$$

Definition 5.**2** An n-dimensional $W_5 -$ Lorentzian-Para-Kenmotsu manifold M is is called

$\boldsymbol{W_5}$ **-semisymmetric** if, for all vector fields X, Y, Z, U, V on M, the following holds:

$$(R(X,Y)\cdot W_5)(U,V)Z = 0. \tag{5.2}$$

Where $R(X,Y)$ serves as a derivation on the curvature tensor W_5.

Theorem 5.1: Any Lorentzian Para-Kenmotsu manifold is **s**emisymmetric.

Theorem 5.2 Any W_5-Lorentzian Para-Kenmotsu manifold is W_5-semisymmetric.

Proof of Theorems
Proof of Theorem 5.1
Proving that

$$(R(X,Y)\cdot R)(U,V)Z = 0. \tag{5.3}$$

Hence, expanding $(R(X,Y)\cdot R)(U,V)Z$ we get

$$(R(X,Y)\cdot R)(U,V)Z = R(X,Y)R(U,V)Z - R(R(X,Y)U,V)Z - R(U,R(X,Y)V)Z - R(U,V)R(X,Y)Z \tag{5.4}$$

Putting $U = \xi$ in (5.4) above gives

$$(R(X,Y)R)\cdot R)(\xi,V)Z = R(X,Y)R(\xi,V)Z - R(R(X,Y)\xi,V)Z - R(\xi,R(X,Y)V)Z - R(\xi,V)R(X,Y)Z \tag{5.5}$$

Simplifying each of the four terms in (5.5) separately yields
First term: $R(X,Y)R(\xi,V)Z$

$$R(X,Y)R(\xi,V)Z = R(X,Y)[g(V,Z)\xi - \eta(Z)V]$$

$$R(X,Y)R(\xi,V)Z = [g(Y,g(V,Z)\xi)X - g(X,g(V,Z)\xi)Y - \eta(Z)g(Y,V)X + \eta(Z)g(X,V)Y$$

$$R(X,Y)R(\xi,V)Z = \eta(Y)g(V,Z)X - \eta(X)g(V,Z)Y - \eta(Z)g(Y,V)X + \eta(Z)g(X,V)Y \tag{5.6}$$

Second Term: $R(R(X,Y)\xi,V)Z$

$$R(R(X,Y)\xi,V)Z = R([\eta(Y)X - \eta(X)Y],V)Z$$

$$= g(V,Z)\eta(Y)X - g(V,Z)\eta(X)Y - g([\eta(Y)X - \eta(X)Y],Z)V$$

$$\therefore R(R(X,Y)\xi,V)Z = \eta(Y)g(V,Z)X - \eta(X)g(V,Z)Y - \eta(Y)g(X,Z)V + \eta(X)g(Y,Z)V \tag{5.7}$$

Third term: $R(\xi,R(X,Y)V)Z$

$$R(\xi,R(X,Y)V)Z = R(\xi,[g(Y,V)X - g(X,V)Y])Z$$

$$= g([g(Y,V)X - g(X,V)Y],Z)\xi - \eta(Z)g(Y,V)X + \eta(Z)g(X,V)Y$$

$$\therefore R(\xi,R(X,Y)V)Z = g(X,Z)g(Y,V)\xi - g(Y,Z)g(X,V)\xi - \eta(Z)g(Y,V)X + \eta(Z)g(X,V)Y \tag{5.8}$$

Fourth term: $R(\xi,V)R(X,Y)Z$

$$R(\xi,V)R(X,Y)Z = g(V,R(X,Y)Z)\xi - \eta(R(X,Y)Z)V$$

$$= g(V,[g(Y,Z)X - g(X,Z)Y])\xi - \eta([g(Y,Z)X - g(X,Z)Y])V$$

$$\therefore R(\xi,V)R(X,Y)Z = g(V,X)g(Y,Z)\xi - g(V,Y)g(X,Z)\xi - \eta(X)g(Y,Z)V + \eta(Y)g(X,Z)V \tag{5.9}$$

Putting equations (5.6), 5.7), (5.8) and (5.9) in equation 5.5) gives

$$R(X,Y)R(\xi,V)Z = \eta(Y)g(V,Z)X - \eta(X)g(V,Z)Y - \eta(Z)g(Y,V)X + \eta(Z)g(X,V)Y \tag{5.6}$$

$$:\therefore R(R(X,Y)\xi,V)Z = \eta(Y)g(V,Z)X - \eta(X)g(V,Z)Y - \eta(Y)g(X,Z)V + \eta(X)g(Y,Z)V \tag{5.7}$$

$\therefore R(\xi, R(X,Y)V)Z = g(X,Z)g(Y,V)\xi - g(Y,Z)g(X,V)\xi - \eta(Z)g(Y,V)X + \eta(Z)g(X,V)Y$ (5.8)

$\therefore R(\xi, V)R(X,Y)Z = g(V,X)g(Y,Z)\xi - g(V,Y)g(X,Z)\xi - \eta(X)g(Y,Z)V + \eta(Y)g(X,Z)V$ (5.9)

$\therefore (R(X,Y)\cdot R)(\xi, V)Z = 0$ (5.10)

This completes the proof that Lorentzian Para-Kenmotsu is a semisymmetric manifold.

Proof Theorem 5.2
Consider

$R(X,Y)W_5(U,W)Z = 0$ (5.11)

$W_5(X,Y)\xi = \eta(Y)X - \eta(X)Y + \frac{1}{n-1}(g(X,\xi)Y - S(X,\xi)Y$ (5.12)
Setting $X = \xi$

$W_5(\xi,Y)\xi = \eta(Y)\xi - \eta(\xi)Y + \frac{1}{n-1}(g(\xi,\xi)Y - S(\xi,\xi)$ (5.13)

$W_5(\xi,Y)\xi = \eta(Y)\xi - \eta(\xi)Y$

For semisymmetric,

$R(X,Y)W_5(U,W)Z = R(X,Y)W_5(U,W)Z - W_5R(X,Y)U,W)Z - W_5(U,R(X,Y)W)Z - W_5(U,W)R(X,Y)Z = 0$ (5.14)

For any $X, Y, U, W, Z \in x(M)$

Taking $X = Z = \xi$ and applying (5.14)

$R(X,Y)\xi = \eta\,(Y)X - \eta(X)Y$ (5.15)

$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X$ (5.16)

Expanding each term in (5.14)

First term $R(X,Y)W_5(U,W)Z$

$R(X,Y)W_5(U,W)Z = g(Y, W_5(U,W)Z)X - g(X, W_5(U,W)Z)Y$ (5.17)

But

$W_5(U,W)Z = g(W,Z)U - g(U,Z)W$

$W_5(U,W)Z = g(Y, g(W,Z)U - g(U,Z)W)X - g(X, g(W,Z)U - g(U,Z)W)Y$ (5.18)

$R(X,Y)W_5(U,W)Z = g(Y,U)g(W,Z) - g(Y,W)g(U,Z)X - g(X,U)g(W,Z)Y - g(X,W)g(U,Z)Y$ (5.19)

Second term $W_5R(X,Y)U,W)Z$

$R(X,Y)U = t = g(Y,U)X - g(X,U)Y$

$W_5R(X,Y)U,W)Z = W_5(t,W)Z = g(W,Z)t - g(t,Z)W$

$W_5R(X,Y)U,W)Z = g(W,Z)[g(Y,U)X - g(X,U)Y] - g([g(Y,U)X - g(X,U)Y],Z)W$

$gW_5R(X,Y)U,W)Z = (W,Z)g(Y,U)X - g(W,Z)g(X,U)Y - g(X,Z)g(Y,U)W + g(Y,Z)g(X,U)W$ (5.20)

Third term $W_5(U,R(X,Y)W)Z$

$W_5(U,R(X,Y)W)Z = W_5(U,m)Z = g(m,Z)U - g(U,Z)m$

$R(X,Y)W = g(Y,W)X - g(X,W)Y$

$g([g(Y,W)X - g(X,W)Y], Z)U - g(U,Z)[g(Y,W)X - g(X,W)Y]$

$$W_5(U, R(X,Y)W)Z = g(X,Z)[g(Y,W)U - g(Y,Z)g(X,W)U - g(U,Z)g(Y,W)X + g(U,Z)g(X,V)Y] \quad (5.21)$$

Forth term $W_5(U,W)R(X,Y)Z$

$R(X,Y)Z = f = g(Y,Z)X - g(X,Z)Y$

$W_5(U,W)R(X,Y)Z = g(W,f)U - g(U,f)W$

$W_5(U,W)R(X,Y)Z = g(W,[g(Y,Z)X - g(X,Z)Y])U - g(U,[g(Y,Z)X - g(X,Z)Y])W$

$$W_5(U,W)R(X,Y)Z = g(W,X)g(Y,Z)U - g(W,Y)g(X,Z)U - g(U,X)g(Y,Z)W - g(U,Y)g(X,Z)W \quad (5.22)$$

Combining (5.19)(5.20),(5.21) and (5.22) additionally, set $U = \xi$

$R(X,Y)W_5(\xi,W)Z - W_5R(X,Y)\xi,W)Z - W_5(\xi,R(X,Y)W)Z - W_5(\xi,W)R(X,Y)Z$

$= [g(Y,\xi)g(W,Z) - g(Y,W)g(\xi,Z)X - g(X,\xi)g(W,Z)Y - g(X,W)g(\xi,Z)Y] - [g(W,Z)g(Y,\xi)X - g(W,Z)g(X,\xi)Y - g(X,Z)g(Y,\xi)W$

$+g(Y,Z)g(X,\xi)W] - [g(X,Z)[g(Y,W)U\xi - g(Y,Z)g(X,W)\xi - g(\xi,Z)g(Y,W)X + g(\xi,Z)g(X,V)Y] - [g(W,X)g(Y,Z)\xi - g(W,Y)g(X,Z)\xi$

$-g(U,X)g(Y,Z)W + g(\xi,Y)g(X,Z)W] = 0$

Hence theorem proved.

6. W_5-Para Kenmotsu Manifold satisfying the condition $W_5 \cdot R = 0$

Definition 6.1 An n-dimensional $W_5 -$ Lorentzian-Para-Kenmotsu manifold M is said to satisfy $W_5 . R = 0$ condition if

$$(W_5(U,V) \cdot R)(X,Y)Z = 0 \quad (6.1)$$

For any vector fields X, Y, Z, U, V on M.

Theorem 6.1, A W_5 $-$Lorentzian-para-Kenmotsu manifold satisfies the condition $W_5 \cdot R = 0$

Corollary 6.2: $W_5 \cdot R -$ Lorentzian-para-Kenmotsu manifold is a $W_5 -$ semisymmetric manifold as well as a semisymmetric manifold (where the W_5-curvature tensor acts as a derivation on the Riemann curvature tensor R)

Proof of theorem 6.1
Proving that the relation (6.1) holds for a W_5 $-$Lorentzian para-Kenmotsu Manifold

The above equation (6.1) can be written as follows:

$(W_5(U,V) \cdot R)(X,Y)Z = W_5(U,V)R(X,Y)Z - R(W_5(U,V)X,Y)Z - R(X,W_5(U,V)Y)Z$

$$-R(X,Y)W_5(U,V)Z \quad (6.2)$$

Putting $U = \xi$ in (6.2) gives

$$(W_5(\xi,V) \cdot R)(X,Y)Z = W_5(\xi,V)R(X,Y)Z - R(W_5(\xi,V)X,Y)Z - R(X,W_5(\xi,V)Y)Z - R(X,Y)W_5(\xi,V)Z \quad (6.3)$$

But
$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y$

$W_5(\xi,V)W = g(V,W)\xi - g(\xi,W)V$

$$W_5(\xi,V)W = g(V,W)\xi - \eta(W)V \quad (6.4)$$

Computing the four terms in (6.3) separately gives

First term: $W_5(\xi,V)R(X,Y)Z$
From (6,4), gives

$W_5(\xi,V)W = [g(V,[g(Y,Z)X - g(X,Z)Y])\xi - \eta([g(Y,Z)X - g(X,Z)Y])V]$

$$W_5(\xi,V)R(X,Y)Z = g(V,X)g(Y,Z)\xi - g(V,Y)g(X,Z)\xi - \eta(X)g(Y,Z)V - \eta(Y)g(X,Z)V \quad (6.5)$$

Second Term: $R(W_5(\xi,V)X,Y)Z$
$W_5(\xi,V)X = g(V,X)\xi - g(\xi,X)V = t$

$R(t,Y)Z = g(Y,Z)t - g(t,Z)Y$

$R([g(V,X)\xi - g(\xi,X)V],Y)Z = g(Y,Z)[g(V,X)\xi - g(\xi,X)V] - g([g(V,X)\xi - g(\xi,X)V],Z)Y$

$R([g(V,X)\xi - g(\xi,X)V],Y)Z = g(Y,Z)g(V,X)\xi - g(Y,Z)g(\xi,X)V - g(\xi,Z)g(V,X)Y + g(V,Z)g(\xi,X)Y$

$W_5(\xi,V)R(X,Y)Z = g(Y,Z)g(V,X)\xi - \eta(X)g(Y,Z)V - \eta(Z)g(V,X)Y + \eta(X)g(V,Z)Y$ (6.6)

Third Term: $R(X,W_5(\xi,V)Y)Z$

$R(X,W_5(\xi,V)Y)Z = g(W_5(\xi,V)Y,Z)X - g(X,Z)W_5(\xi,V)Y$

$R(X,W_5(\xi,V)Y)Z = g([g(V,Y)\xi - \eta(Y)V],Z)X - g(X,Z)[g(V,Y)\xi - \eta(Y)V]$

$R(X,W_5(\xi,V)Y)Z = \eta(Z)g(V,Y)X - \eta(Y)g(V,Z)X - g(X,Z)g(V,Y)\xi + \eta(Y)g(X,Z)V]$ (6.7)

Fourth Term: $R(X,Y)W_5(\xi,V)Z$

$W_5(\xi,V)Z = g(V,Z)\xi - \eta(Z)V$

$R(X,Y)W_5(\xi,V)Z = g(Y,[g(V,Z)\xi - \eta(Z)V])X - g(X,[g(V,Z)\xi - \eta(Z)V])Y$

$R(X,Y)W_5(\xi,V)Z = \eta(Y)g(V,Z)X - \eta(Z)g(Y,V)X - \eta(X)g(V,Z)Y + \eta(Z)g(X,V)Y$ ` (6.8)

Plugging in equations (6.5), (6.6), (6.7) and (6.8) in (6.3) gives

$\therefore W_5 \cdot R = 0$ (6.9)

This completes the proof of the theorem (6.9)

7. A $\phi - W_5$-LP-Kenmotsu manifold

Definition 7.1: An n-dimensional LP-Kenmotsu manifold is said to be $\phi - W_5$-flat if
W_5 -curvature tensor satisfies the following condition

$W_5(X,Y).\phi)Z = W_5(X,Y)\phi Z - \phi(W_5(X,Y)Z = 0$ (7.1)

Theorem 7: A W_5 -LP-Kenmotsu -manifold is a $\phi - W_5\ flat$ manifold
Proof.
Consider $\phi - W_5$ LP-Kenmotsu manifold, then the following hold

$W_5(X,Y)\phi Z - \phi(W_5(X,Y)Z$

$= R(X,Y)\phi Z - g(Y,Z)\phi X + g(X,Z)\phi Y + g(X,Y)\phi Z - g(Y,\phi Z)X - g(X,Y)\phi Z + g(Y,Z)\phi X$ (7.2)
$= R(X,Y)\phi Z + g(X,Z)\phi Y - g(Y,\phi Z)X$

$R(X,Y)\phi Z = g(Y,\phi Z)X - g(X,Z)\phi Y$ (7.3)

$= g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y + g(X,Y)\phi Z - g(Y,\phi Z)X - g(X,Y)\phi Z + g(Y,Z)\phi X$ (7.4)

$= -g(X,\phi Z)Y + g(X,Z)\phi Y$ (7.5)

Putting $Y = Z = \xi$ in above equation yields

$W_5(X,\xi).\phi)\xi = 0$

This completes the proof.

8. A $W_5 - Q$ L.P-Kenmotsu manifold
Definition; W_5 LP Kenmotsu manifold is such that

$W_5.Q = 0$

Theorem 8.1; An n-dimension L.P Kenmotsu manifold satisfy the condition

$W_5 . Q = 0$

Proof;
Consider $W_5 . Q = 0$
Therefore

$W_5(X, Y)QZ = W_5(X, Y)QZ - Q(W_5)(X, Y)Z$ (8.1)

$= R(X, Y)QZ - Q(R(X, Y)Z - \frac{1}{n-1}[S(X, Z)QY - g(X, QZ)QY] + \frac{1}{n-1}[S(X, Z)QY - Q(g(X, Z)QY]$ (8.2)

$= R(X, Y)QZ - Q(R(X, Y)Z) - \frac{1}{n-1}(S(X, Z)QY - S(X, Z)QY - (g(X, QZ)QY + Q(g(X, Z)QY)$ (8.3)

$R(X, Y)QZ - Q(R(X, Y)Z - \frac{1}{n-1}[Q(g(X, Z)QY - g(X, QZ)QY]$ (8.4)

But
$R(X, Y)QZ = g(Y, QZ)X - g(X, QZ)Y$

$[g(Y, QZ)X - g(X, QZ)Y] - [g(Y, Z)QX - g(X, Z)QY] - \frac{1}{n-1}[Q(g(X, Z)QY - g(X, QZ)QY]$
(8.5)

$= [g(Y, QZ)X - g(X, QZ)Y] - [g(Y, Z)QX - g(X, Z)QY] - [Q(g(X, Z)QY - g(X, QZ)Y]$
(8.6)

$= g(Y, QZ)X - g(Y, Z)QX$

$= (n-1)g(Y, Z)X - (n-1)g(Y, Z)X$

$= 0$

9. An n-dimensional L.P Kenmotsu manifold satisfying $Q - W_5 = 0$

Definition 9.1; An n-dimensional L. P Kenmotsu manifold is said to satisfy the condition

$Q - W_5 = 0$ if $Q . W_5(X, Y)Z = 0$.
Theorem 9.1; A Lorentzian para kenmotsu manifold satisfying the condition $Q . W_5 = 0$ is a special type of $\eta -$Einstein manifold.

Considering the Lorentzian Para Kenmotsu manifold which satisfies the condition

$Q . W_5(X, Y)Z = 0$

Therefore

$Q . W_5(X, Y)Z - W_5(QX, Y)Z - W_5(X, QY)Z - W_5(X, Y)QZ = 0$ (9.1)

Computing the four terms separately

First term
$Q(W_5(X, Y)Z = g(Y, Z)QX - g(X, Z)QY - \frac{1}{n-1}[S(X, Z)QY - Qg(X, Z)QY]$ (9.2)

$= g(Y, Z)QX - g(X, Z)QY - g(X, Z)QY + g(X, Z)QY$ (9.3)

$Q(W_5(X, Y)Z = g(Y, Z)QX - g(X, Z)QY$ (9.4)

Second term

$W_5(QX, Y)Z = g(Y, Z)QX - g(QX, Z)Y - g(QX, Z)Y$ (9.5)

$= g(Y, Z)QX) - g(QX, Z)Y - g(QX, Z)Y + g(X, Z)QY - g(X, Z)QY$ (9.6)

$W_5(QX, Y)Z = g(Y, Z)QX - 2g(QX, Z)Y + g(X, Z)QY$ (9.7)

Third term

$W_5(X,QY)Z = g(QY,Z)X - g(X,Z)QY - [g(X,QZ)Y - g(QX,Z)Y]$ (9.8)

$= g(Y,QZ)X - g(X,Z)QY$ (9.9)

Fourth term

$W_5(X,Y)QZ$

$W_5(X,Y)QZ = g(Y,QZ)X - g(X,QZ)Y - [g(X,Z)QY - g(X,QZ)Y]$ (9.10)

$= g(Y,QZ)X - g(X,Z)QY$ (9.11)

Combining the four terms

$g(Y,Z)QX - g(X,Z)QY] - g(Y,Z)QX - 2g(QX,Z)Y + g(X,Z)QY - [g(Y,QZ)X - g(X,Z)QY] - [g(Y,QZ)X - g(X,Z)QY]$ (9.12)

$= 2g(QX,Z)Y - g(Y,QZ)X - g(Y,QZ)X$

$= 2[g(QX,Z)Y - g(Y,QZ)X]$

$= 2[g(QX,Z)Y - g(Y,QZ)X] = 0$ (9.13)

Let $Y = \xi$ and taking inner product of ξ gives

$2[\mathrm{g}(\mathrm{X},\mathrm{Z})\xi - \mathrm{g}(\xi,\mathrm{Z})\mathrm{X}] = 0$ (9.14)

But

$S(X,\xi)Y = (\mathrm{n}-1)\mathrm{g}(\mathrm{X},\xi)\mathrm{Y}$ (9.15)

And

$g(X,Z)\xi = \eta(X)\eta(Z)$ (9.16)

Therefore

$g(X,\xi)Y = \frac{S(X,\xi)Y}{n-1}$ (9.17)

$\frac{S(X,\xi)Y}{n-1} = \eta(\mathrm{X})\eta(\mathrm{Y})$ (9.18)

$S(X,\xi)Y = (n-1)\eta(X)\eta(Y)$ (9.19)

10.0. An n-dimensional LP-Kenmotsu manifold satisfying the condition $W_5.W_5 = 0$

Definition 10.1 An n-dimensional LP-Kenmotsu manifold is said to satisfy the condition $W_5.W_5 = 0$ if

$W_5.W_5 = W_5(U,V).W_5)(X,Y)Z = 0$

Theorem 10.1: An LP-Kenmotsu manifold satisfying the condition $W_5.W_5 = 0$ is a flat manifold
Proof:

Consider $W_5.W_5 = W_5(U,V).W_5)(X,Y)Z = 0$

Where,

$W_5(U,V).W_5)(X,Y)Z = W_5(U,V)W_5(X,Y)Z - W_5(W_5(U,V)X,Y)Z - W_5(X,W_5(U,V)Y)Z$

$-W_5(X,Y)W_5(U,V)Z = 0$ (10.1)

Putting $V = \xi$ in (10.1) above we get

$W_5(U,\xi)W_5(X,Y)Z - W_5(W_5(U,\xi)X,Y)Z - W_5(X,W_5(U,\xi)Y)Z$

$-W_5(X,Y)W_5(U,\xi)Z = 0$ (10.2)

Simplifying each term in (10.2) separately gives

Term 1: $W_5(U,\xi)W_5(X,Y)Z$

Let $W_5(X,Y)Z = W = g(Y,Z)X - g(X,Z)Y$

$R(U,\xi)W = g(\xi,W)U - g(U,W)\xi$

$W_5(U,\xi)W = g(\xi,W)U - g(U,W)\xi + \frac{1}{n-1}[g(U,W)Q\xi - S(U,W)\xi]$

$W_5(U,\xi)W = g(\xi,W)U - g(U,W)\xi$

$W_5(U,\xi)W = g(\xi, g(Y,Z)X - g(X,Z)Y), U - g(U,[g(Y,Z)X - g(X,Z)Y])\xi$

$W_5(U,\xi)W = g(\xi,X)g(Y,Z)U - g(\xi,Y)g(X,Z)U - g(U,X)g(Y,Z)\xi + g(U,Y)g(X,Z)\xi)$ $W_5(U,\xi)W = \eta(X)g(Y,Z)U - \eta(Y)g(X,Z)U - g(U,X)g(Y,Z)\xi + g(U,Y)g(X,Z)\xi)$ (10.3)

Term II: $W_5(W_5(U,\xi)X,Y)Z$
Applying the definition of W_5 on above, we get

$W_5(W_5(U,\xi)X,Y)Z = g(Y,Z)W_5(U,\xi)X - g(W_5(U,\xi)X,Z)Y + \frac{1}{n-1}[g(W_5(U,\xi)X,Z)QY - S(W_5(U,\xi)X,Z)Y]$

$W_5(W_5(U,\xi)X,Y)Z = W_5(f,Y)Z = g(Y,Z)f - g(f,Z)Y$

$W_5(U,\xi)X = f = g(\xi,X)U - g(U,X)\xi$

$W_5(W_5(U,\xi)X,Y)Z = g(Y,Z)[g(\xi,X)U - g(U,X)\xi] - g([g(\xi,X)U - g(U,X)\xi],Z)Y$

$W_5(W_5(U,\xi)X,Y)Z = g(Y,Z)g(\xi,X)U - g(Y,Z)g(U,X)\xi - g(U,Z)[g(\xi,X)Y - g(\xi,Z)g(U,X)Y$

$W_5(W_5(U,\xi)X,Y)Z = g(Y,Z)\eta(X)U - g(Y,Z)g(U,X)\xi - g(U,Z)\eta(X)Y + \eta(Z)g(U,X)Y$ (10.4)

Term III: $W_5(X,W_5(U,\xi)Y)Z$

$W_5(X,W_5(U,\xi)Y)Z = R(X,W_5(U,\xi)Y)Z$

$W_5(U,\xi)Y = t = g(\xi,Y)U - g(U,Y)\xi$

$R(X,W_5(U,\xi)Y)Z = R(X,t)Z = g(t,Z)X - g(X,Z)t$

$R(X,W_5(U,\xi)Y)Z = R(X,[g(\xi,Y)U - g(U,Y)\xi])Z = g([g(\xi,Y)U - g(U,Y)\xi],Z)X - g(X,Z)[g(\xi,Y)U - g(U,Y)\xi]$

$R(X,W_5(U,\xi)Y)Z = g(U,Z)(g(\xi,Y)X - g(\xi,Z)g(U,Y)X - g(X,Z)g(\xi,Y)U + g(X,Z)g(U,Y)\xi$

$R(X,W_5(U,\xi)Y)Z = g(U,Z)\eta(Y)X - \eta(Z)g(U,Y)X - g(X,Z)\eta(Y)U + g(X,Z)g(U,Y)\xi$ *(10.5)*

Term IV: $W_5(X,Y)W_5(U,\xi)Z$

$W_5(X,Y)W_5(U,\xi)Z = R(X,Y)W_5(U,\xi)Z$

$W_5(U,\xi)Z = B = g(\xi,Z)U - g(U,Z)\xi$

$W_5(X,Y)B = g(Y,B)X - g(X,B)Y$

$W_5(X,Y)[g(\xi,Z)U - g(U,Z)\xi] = g(Y,[g(\xi,Z)U - g(U,Z)\xi])X - g(X,[g(\xi,Z)U - g(U,Z)\xi])Y$

$W_5(X,Y)W_5(U,\xi)Z = g(Y,U)g(\xi,Z)X - g(Y,\xi)g(U,Z)X - g(X,U)g(\xi,Z)Y - g(X,\xi)g(U,Z)Y$

$W_5(X,Y)W_5(U,\xi)Z = g(Y,U)\eta(Z)X - \eta(Y)g(U,Z)X - g(X,U)\eta(Z)Y + \eta(X)g(U,Z)Y$ (10.6)

Combing the four terms

First term $W_5(U,\xi)W = \eta(X)g(Y,Z)U - \eta(Y)g(X,Z)U - g(U,X)g(Y,Z)\xi + g(U,Y)g(X,Z)\xi)$

Second term $W_5(W_5(U,\xi)X,Y)Z = g(Y,Z)\eta(X)U - g(Y,Z)g(U,X)\xi - g(U,Z)\eta(X)Y + \eta(Z)g(U,X)Y$

Third term $R(X,W_5(U,\xi)Y)Z = g(U,Z)\eta(Y)X - \eta(Z)g(U,Y)\,X - g(X,Z)\eta(Y)U + g(X,Z)g(U,Y)\xi$

Fourth term $W_5(X,Y)W_5(U,\xi)Z = g(Y,U)\eta(Z)X - \eta(Y)g(U,Z)X - g(X,U)\eta(Z)Y + \eta(X)g(U,Z)Y$

Therefore,
$W_5(U,\xi)W - W_5(W_5(U,\xi)X,Y)Z - R(X,W_5(U,\xi)Y)Z - W_5(X,Y)W_5(U,\xi)Z = 0$

This completes proof of the theorem that an LP-Kenmotsu manifold satisfying $W_5 \cdot W_5 = 0$ is a flat manifold

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