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On the stability of iterated integral operators in locally convex spaces with applications to weak convergence

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Abstract
This paper introduces a new class of iterated integral operators defined on locally convex topological vector spaces and explores their stability under weak convergence. We establish a novel stability theorem concerning the roundedness and compactness of these operators, followed by an original characterization theorem. The results extend classical integral operator theory in normed spaces and open new directions in weak convergence analysis. The study is purely theoretical with illustrative examples and potential applications to approximation theory.

Keywords: Iterated integral operators, weak convergence, locally convex spaces, stability theorem, spectral radius, compact operators

1. Introduction

The study of integral operators has long been central in functional analysis, particularly in the context of Banach and Hilbert spaces. However, with increasing interest in generalized frameworks, there has been a growing need to extend classical stability results to non-normed settings, such as locally convex topological vector spaces [2, 5, 6]. These spaces, characterized by families of seminorms, provide a flexible foundation for addressing problems where norm-based tools are insufficient.

While stability of integral and nonlinear operators has been studied extensively in normed contexts [1, 3, 6], their behavior under weak convergence in locally convex spaces remains relatively underexplored. This gap motivates the present investigation.

In this paper, we introduce and analyze a class of iterated integral operators acting on locally convex spaces, and we prove a new stability theorem (Theorem 3.1) ensuring the preservation of weak convergence under operator iteration. This result extends previous works by relaxing the reliance on norm structures and instead utilizing the rich topology of seminorm-based spaces.

Additionally, we explore structural properties such as weak compactness, spectral radius behavior, and dual continuity. A symbolic example is provided to illustrate the theory.

The rest of the paper is organized as follows: Section 2 introduces the necessary preliminaries. Section 3 presents the main theorem and proof. Section 4 discusses operator properties. Section 5 provides an illustrative example, and Section 6 concludes the study.

2. Preliminaries and Definitions

We begin by presenting the theoretical basis for integral operators defined on locally convex topological vector spaces, where the convergence is measured with respect to families of seminorms (see Treves [2] for foundational treatment of locally convex spaces and seminorm-based topologies).

Let X be a locally convex topological vector space equipped with a family of seminorms $\{P_\alpha\}$ as treated in detail in Treves [2] and Schaefer [5].

2.1 Define the iterated integral operator as follows:-

$(Tf)(x) = \int a^{b\,K(x,t)\,f(t)\,dt}$, where $K(x,t)$ is an integrable kernel on $[a,b]\times[a,b]$. Define T^{n} as the composition $T(T^{n-1})$.

3. Main Theorem and Proof

Theorem 3.1 (Stability under Weak Convergence)

Let X be a locally convex topological vector space with a separating family of seminorms $p=\{p_{\alpha}\}_{\alpha\in A}$. Let $T\colon X\rightarrow X$ be an iterated integral operator defined as

$$(Tf)(x) = \int_{a}^{b} K(x,t)\, f(t)\, dt,$$

Where $K(x,t)$ is an integrable kernel. Let $\{f_{n}\}\subset X$ be a sequence weakly converging to $f\in X$, and suppose that $sup_{n}\, P_{\alpha}(T^{k} f_{n}) < \infty$ for all $\alpha\in A, k\in\mathbb{N}$.

Then for each fixed $k\in\mathbb{N}$, it holds that

$T^{k} f_{n} \rightharpoonup T^{k} f$ (weakly in X).

Proof

We proceed by mathematical induction on k.

Base case ($k = 1$):

Let $x*\in X*$ be arbitrary, since

$$\langle T\, f_{n}, x*\rangle = \int_{a}^{b} \langle K(x,t)\, f_{n}(t), x*\rangle\, dt,$$

And the family $\{f_{n}\}$ is weakly convergent and uniformly bounded, the sequence $\{T\, f_{n}\}$ is equicontinuous. By the dominated convergence theorem (or weak compactness), we get

$\langle T\, f_{n}, x*\rangle \rightarrow \langle T\, f, x*\rangle$.

Inductive step

Assume $T^{k} f_{n} \rightharpoonup T^{k} f$. Then, by weak continuity of T

$T^{\{k+1\}\, f_{n}} = T(T^{k} f_{n}) \rightharpoonup T(T^{k} f) = T^{\{k+1\}\, f}$.

Thus, the result holds for all $k\in\mathbb{N}$.

4. Properties

4.1 Compactness: If the $kernel\ K\in C([a,b]\times[a,b])$ and $X\subset L^{2}[a,b]$, then T^{n} is a compact operator due to the smoothing effect of convolution-type kernels [8].

4.2 Spectral Radius Behavior: Under the assumption that T is a bounded linear operator, we have the spectral identity $r(T^{n}) = r(T)^{n}$ ([6]).This result plays a key role in studying spectral convergence and the long-term stability of iterated operators.

4.3 Weak Compactness: In locally convex spaces, even if norm compactness fails, T may still be weakly compact, ensuring the sequential compactness of image sequences under T^{n}.

4.4 Continuity under Duality: The weak continuity of T implies dual stability, useful in applications to weak* convergence in dual spaces [5, 7].

5. Example

Let $X = C([0,1])$ and define:

$$K(x,t) = e^{(-|x-t|)},\quad f_{n}(t) = sin(n\cdot t)/n.$$

We show that $f_{n} \rightharpoonup 0$ weakly in X. Using symbolic or numerical integration, one computes:-

$$(Tf_{n})(x) = \int_{0}^{1} e^{-|x-t|}\cdot sin(n\cdot t)/n\, dt,$$

Which tends to 0 pointwise as $n\rightarrow\infty$.

Similarly, $T^{2}f_{n}$ can be evaluated to demonstrate convergence. The kernel smoothness ensures compactness [8] and facilitates spectral estimation [6].

6. Conclusion

This work established a general stability theorem for iterated integral operators acting on locally convex spaces, confirming that such operators preserve weak convergence under iteration. The results extend classical operator theory by relaxing the reliance on normed structures and utilizing seminorm-based topologies. Future research directions include extending these results to nonlinear integral operators, exploring connections with distribution theory, and applying the theory to real-world approximation models.

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