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Anooja I
Assistant Professor, Department of Mathematics, CMS College, Kottayam (Autonomous), Affiliated to Mahatma Gandhi University, Kottayam, Kerala, India

Corresponding Author:
Anooja I
Assistant Professor, Department of Mathematics, CMS College, Kottayam (Autonomous), Affiliated to Mahatma Gandhi University, Kottayam, Kerala, India

Codes and Graphs

Anooja I

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Abstract
This paper explores the construction of codes derived from graphs, a rapidly evolving area intersecting graph theory and coding theory. The study examines various families of graphs to illustrate their potential for generating codes. Furthermore, connections between graph invariants and code properties are investigated to provide a theoretical foundation and guide the design of future graph-based codes.

Keywords: Graphs, codes, vertex polynomials, neighbourhood polynomials

Introduction

Codes from graphs form an important link between graph theory and coding theory. In this approach, codes are constructed by associating graph parameters -such as neighbourhood polynomials and vertex polynomials. The topology of a graph influences key code parameters, enabling the design of efficient, reliable codes for communication and data storage. This method leverages the combinatorial properties of graphs to create codes with good distance, dimension, and decoding performance, making it a valuable tool for modern information theory and network applications.

1. Codes from graphs

(i) Using Neighbourhood Polynomials

Let *G* be any simple graph. Consider the neighbourhood complex N (*G*) of the graph *G*. Form the neighbourhood polynomial of graph *G*. Change all the coefficients of this polynomial as binary digits by changing all non-zero coefficients as 1s and denote this polynomial as $m(x)$. Take all the coefficients of this polynomial $m(x)$ as the input stream *m*.

Consider all the subgraphs of *G*. Form the neighbourhood polynomials corresponding to each subgraph H_i of *G*. Change all the coefficients of these polynomials as binary digits by changing all non-zero coefficients as 1s and denote these polynomials as $g_i(x)$. The output streams c_i are obtained from the relation $c_i(x) = m(x) g_i(x)$. The output streams together will produce the coded stream *C*.

Example 1: Consider the graph C_4.

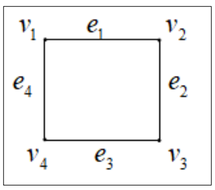

Fig 1" - C_4

N (C_4) = $\{\phi,\{v_1\},\{v_2\},\{v_3\},\{v_4\},\{v_1,v_3\},\{v_2,v_4\}\}$.

$neigh_{C_4}(x) = 1+4x+2x^2$ and $m(x) = 1+x+x^2$. The input stream is $m = \{1,1,1\}$.

Consider all the subgraphs of C_4.

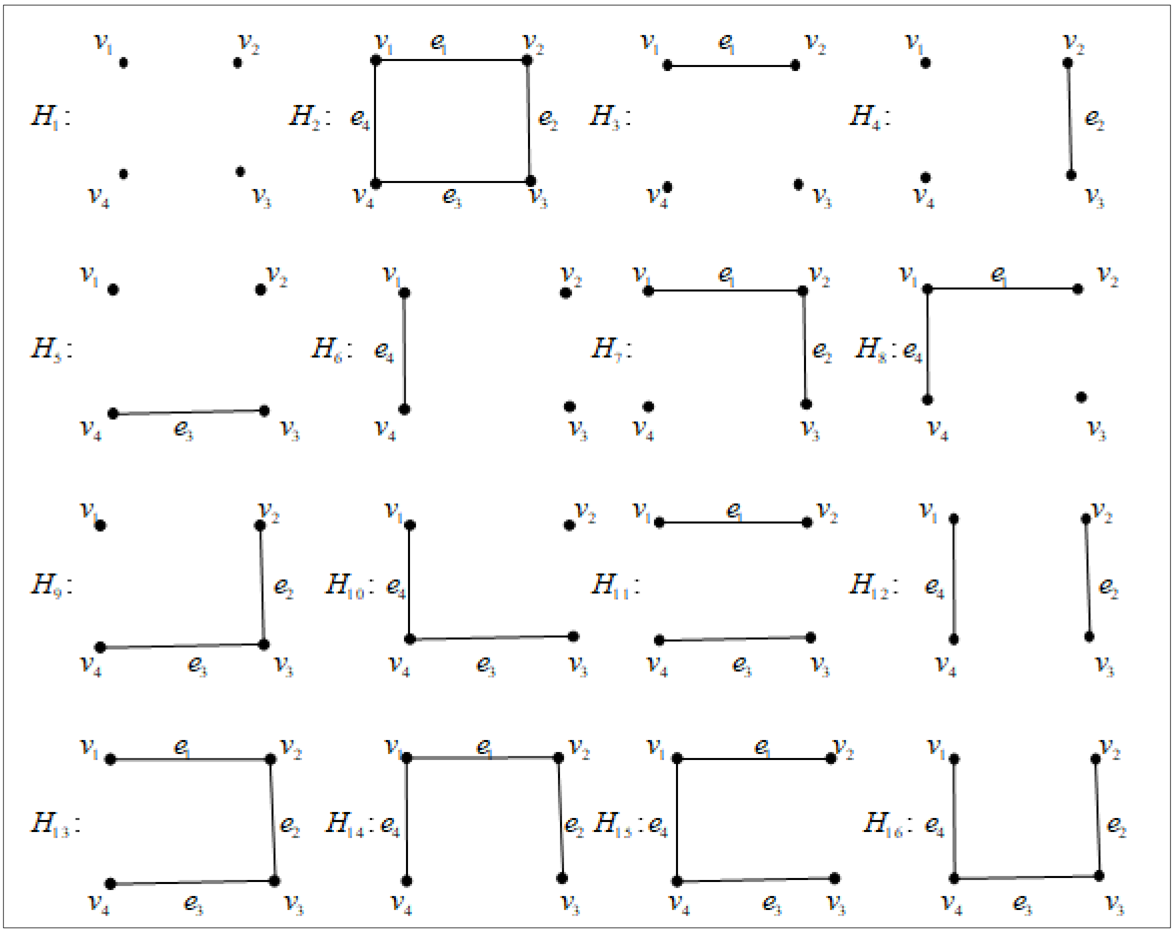

Fig 2

N (H_1) = $\{\phi\}$ $neigh_{H_1}(x) = 1$ $g_1(x) = 1$

N (H_2) = $\{\phi,\{v_1\},\{v_2\},\{v_3\},\{v_4\},\{v_1,v_3\},\{v_2,v_4\}\}$

$neigh_{H_2}(x) = 1+4x+2x^2$ $g_2(x) = 1+x+x^2$

N (H_3) = $\{\phi,\{v_1\},\{v_2\}\}$ $neigh_{H_3}(x) = 1+2x$ $g_3(x) = 1+x$

N (H_4) = $\{\phi,\{v_2\},\{v_3\}\}$ $neigh_{H_4}(x) = 1+2x$ $g_4(x) = 1+x$

N (H_5) = $\{\phi,\{v_3\},\{v_4\}\}$ $neigh_{H_5}(x) = 1+2x$ $g_5(x) = 1+x$

N (H_6) = $\{\phi,\{v_1\},\{v_4\}\}$ $neigh_{H_6}(x) = 1+2x$ $g_6(x) = 1+x$

N (H_7) = $\{\phi,\{v_1\},\{v_2\},\{v_3\},\{v_1,v_3\}\}$ $neigh_{H_7}(x) = 1+3x+x^2$ $g_7(x) = 1+x+x^2$

N (H_8) = $\{\phi,\{v_1\},\{v_2\},\{v_4\},\{v_2,v_4\}\}$ $neigh_{H_8}(x) = 1+3x+x^2$ $g_8(x) = 1+x+x^2$

N (H_9) = $\{\phi,\{v_2\},\{v_3\},\{v_4\},\{v_2,v_4\}\}$ $neigh_{H_9}(x)=1+3x+x^2$ $g_9(x)=1+x+x^2$

N (H_{10}) = $\{\phi,\{v_1\},\{v_3\},\{v_4\},\{v_1,v_3\}\}$ $neigh_{H_{10}}(x)=1+3x+x^2$ $g_{10}(x)=1+x+x^2$ N

(H_{11}) = $\{\phi,\{v_1\},\{v_2\},\{v_3\},\{v_4\}\}$ $neigh_{H_{11}}(x)=1+4x$ $g_{11}(x)=1+x$

N (H_{12}) = $\{\phi,\{v_1\},\{v_2\},\{v_3\},\{v_4\}\}$ $neigh_{H_{12}}(x)=1+4x$ $g_{12}(x)=1+x$ N

(H_{13}) = $\{\phi,\{v_1\},\{v_2\},\{v_3\},\{v_4\},\{v_1,v_3\},\{v_2,v_4\}\}$ $neigh_{H_{13}}(x)=1+4x+2x^2$ $g_{13}(x)=1+x+x^2$

N (H_{14}) = $\{\phi,\{v_1\},\{v_2\},\{v_3\},\{v_4\},\{v_1,v_3\},\{v_2,v_4\}\}$ $neigh_{H_{14}}(x)=1+4x+2x^2$ $g_{14}(x)=1+x+x^2$

N (H_{15}) = $\{\phi,\{v_1\},\{v_2\},\{v_3\},\{v_4\},\{v_1,v_3\},\{v_2,v_4\}\}$ $neigh_{H_{15}}(x)=1+4x+2x^2$ $g_{15}(x)=1+x+x^2$

N (H_{16}) = $\{\phi,\{v_1\},\{v_2\},\{v_3\},\{v_4\},\{v_1,v_3\},\{v_2,v_4\}\}$ $neigh_{H_{16}}(x)=1+4x+2x^2$ $g_{16}(x)=1+x+x^2$

$c_1(x)=1+x+x^2$ $c_1=\{1,1,1\}$ $c_2(x)=1+x^2+x^4$ $c_2=\{1,0,1,0,1\}$

$c_3(x)=1+x^3$ $c_3=\{1,0,0,1\}$ $c_4(x)=1+x^3$ $c_4=\{1,0,0,1\}$

$c_5(x)=1+x^3$ $c_5=\{1,0,0,1\}$ $c_6(x)=1+x^3$ $c_6=\{1,0,0,1\}$

$c_7(x)=1+x^2+x^4$ $c_7=\{1,0,1,0,1\}$ $c_8(x)=1+x^2+x^4$ $c_8=\{1,0,1,0,1\}$

$c_9(x)=1+x^2+x^4$ $c_9=\{1,0,1,0,1\}$ $c_{10}(x)=1+x^2+x^4$ $c_{10}=\{1,0,1,0,1\}$

$c_{11}(x)=1+x^3$ $c_{11}=\{1,0,0,1\}$ $c_{12}(x)=1+x^3$ $c_{12}=\{1,0,0,1\}$

$c_{13}(x)=1+x^2+x^4$ $c_{13}=\{1,0,1,0,1\}$ $c_{14}(x)=1+x^2+x^4$ $c_{14}=\{1,0,1,0,1\}$

$c_{15}(x)=1+x^2+x^4$ $c_{15}=\{1,0,1,0,1\}$ $c_{16}(x)=1+x^2+x^4$ $c_{16}=\{1,0,1,0,1\}$

The distinct outputs are $111,1001,10101$. Leave the output corresponding to the null graph. Then we get the output streams are 1001 and 10101. The code corresponding to the graph C_4 is $C=\{00,01,10,11\}$.

Similarly, the graphs $C_5,C_6,C_7,\ldots$ can be coded as $C=\{00,01,10,11\}$.

Result 1.1 Any cycle C_n can be coded as $C=\{00,01,10,11\}$ using neighbourhood polynomial.

(ii) Using Vertex Polynomial

Let *G* be any simple graph. Consider the vertex polynomial of graph *G*. Change all the coefficients of this polynomial as binary digits by changing all non-zero coefficients as 1s and denote this polynomial as $m(x)$. Take all the coefficients of this polynomial $m(x)$ as the input stream *m*.

Consider all the subgraphs of *G*. Form the vertex polynomials corresponding to each subgraph H_i of *G*. Change all the coefficients of these polynomials as binary digits by changing all non-zero coefficients as 1s and denote these polynomials as $g_i(x)$. The output streams c_i are obtained from the relation $c_i(x)=m(x)g_i(x)$. The output streams together will produce the coded stream *C*.

Remark The vertex polynomial corresponding to a null graph can be treated as zero.

Definition 1.2 The number of codewords needed to get a cyclic code from the code formed using neighbourhood polynomial (vertex polynomial) is known as the cyclic deficiency.

Example 2 Consider the graph C_4 given in figure 1 and all the subgraphs of C_4 given in figure 2.

$V(C_4;x)=4x^2$. Then $m(x)=x^2$.

Now, $V(H_1;x)=0$ $g_1(x)=0$ $c_1(x)=0$ $c_1=\{0,0,0\}$

$V(H_2;x)=4x^2$ $g_2(x)=x^2$ $c_2(x)=x^4$ $c_2=\{0,0,0,0,1\}$

$V(H_3;x)=2+2x$ $g_3(x)=1+x$ $c_3(x)=x^2+x^3$ $c_3=\{0,0,1,1\}$

$V(H_4;x)=2+2x$ $g_4(x)=1+x$ $c_4(x)=x^2+x^3$ $c_4=\{0,0,1,1\}$

$V(H_5;x)=2+2x$ $g_5(x)=1+x$ $c_5(x)=x^2+x^3$ $c_5=\{0,0,1,1\}$

$V(H_6;x)=2+2x$ $g_6(x)=1+x$ $c_6(x)=x^2+x^3$ $c_6=\{0,0,1,1\}$

$V(H_7;x)=1+2x+x^2$ $g_7(x)=1+x+x^2$ $c_7(x)=x^2+x^3+x^4$ $c_7=\{0,0,1,1,1\}$

$V(H_8;x)=1+2x+x^2$ $g_8(x)=1+x+x^2$ $c_8(x)=x^2+x^3+x^4$ $c_8=\{0,0,1,1,1\}$

$V(H_9;x)=1+2x+x^2$ $g_9(x)=1+x+x^2$ $c_9(x)=x^2+x^3+x^4$ $c_9=\{0,0,1,1,1\}$

$V(H_{10};x)=1+2x+x^2$ $g_{10}(x)=1+x+x^2$ $c_{10}(x)=x^2+x^3+x^4$ $c_{10}=\{0,0,1,1,1\}$

$V(H_{11};x)=4x$ $g_{11}(x)=x$ $c_{11}(x)=x^3$ $c_{11}=\{0,0,0,1\}$

$V(H_{12};x)=4x$ $g_{12}(x)=x$ $c_{12}(x)=x^3$ $c_{12}=\{0,0,0,1\}$

$V(H_{13};x)=2x+2x^2$ $g_{13}(x)=x+x^2$ $c_{13}(x)=x^3+x^4$ $c_{13}=\{0,0,0,1,1\}$

$V(H_{14};x)=2x+2x^2$ $g_{14}(x)=x+x^2$ $c_{14}(x)=x^3+x^4$ $c_{14}=\{0,0,0,1,1\}$

$V(H_{15};x)=2x+2x^2$ $g_{15}(x)=x+x^2$ $c_{15}(x)=x^3+x^4$ $c_{15}=\{0,0,0,1,1\}$

$V(H_{16};x)=2x+2x^2$ $g_{16}(x)=x+x^2$ $c_{16}(x)=x^3+x^4$ $c_{16}=\{0,0,0,1,1\}$

The distinct outputs are 000, 0011, 0001, 00001, 00011, 00111.
These codes can be rewritten as 00000, 00110, 00010, 00001, 00011, 00111 such that all the codewords are of same length. Then the code corresponding to the graph C_4 is $C=\{000,110,010,001,011,111\}$.

Similarly, the graph C_3 can be coded as $C=\{000,110,001,011\}$, C_5, C_7,... are coded as $C=\{000,110,001,011,111\}$ and C_6,C_8,... are coded as $C=\{000,110,010,001,011,111\}$. Now, from the above observations are concluded as follows:

Result 1.3 The graph C_n can be coded with cyclic deficiency 3 if *n* is an odd number greater than or equal to 3 and coded with cyclic deficiency 2 if *n* is an even number greater than 2.

Next, try to code the complete graph K_n using neighbourhood polynomial and vertex polynomial.

(i) Using Neighbourhood Polynomial

Consider K_4. Then corresponding $m(x)=1+x+x^2+x^3$. The possible distinct $g_i(x)$ are $1, 1+x, 1+x+x^2, 1+x+x^2+x^3$. The various outputs are 1111, 10001, 101101, 1010101. Leave the output corresponding to the null graph. Then we get the output streams are 10001, 101101 and 1010101.These can be rewritten as 1000100, 1011010 and 1010101. Hence K_4 can be coded as $C=\{000,001,010,011,101,111\}$.

Consider K_5. Then corresponding $m(x)=1+x+x^2+x^3+x^4$. The possible distinct $g_i(x)$ are $1, 1+x, 1+x+x^2, 1+x+x^2+x^3, 1+x+x^2+x^3+x^4$. The various outputs are 11111, 100001, 1011101, 10100101, 101010101. Leave the output corresponding to the null graph. Then we get the output streams are 100001, 1011101, 10100101 and 101010101. These can be rewritten as 100001000, 101110100, 101001010 and 101010101. Hence K_5 can be coded as $C=\{0000,0001,0010,0100,0101,0111,1010,1111\}$.

Similarly, K_6 can be coded as

$C=\{00000,00001,00010,00101,01010,01000,01111,10101,01011,11111\}$,

K_7 can be coded as $C=\{000000,000001,000010,000101,001010,010100,010101,010000,$

$010111,011111,101010,111111\}$,

K_8 can be coded as $C=\{0000000,0000001,0000010,0000101,0010101,0100000,0101000,0101010,$

$0101011,0101111,011111,0001010,1010101,1111111\}$.

Result 1.4 In K_4, the code formed using neighbourhood polynomial is $\{111,000,011,010,101,001\}$. This code will become a cyclic code if we add two more elements 100 and 110. Hence here the cyclic deficiency is 2. Similarly, in K_5, K_6, K_7, K_8 *and* K_9 the cyclic deficiencies are 4, 12, 16, 30 and 37 respectively.

(ii) Using Vertex Polynomial

Consider K_3. Then corresponding $m(x)=x^2$. The possible distinct $g_i(x)$ are $0, 1+x, x+x^2, x^2$. The various outputs are 000, 0011, 00011, 00001. These can be rewritten as 00000, 00110, 00011, 00001.

Hence K_3 can be coded as $C=\{000,110,001,011\}$.

Consider K_4. Then corresponding $m(x)=x^3$. The possible distinct $g_i(x)$ are $0, x, x^2, x^3, x+x^2, x+x^3, x^2+x^3, x+x^2+x^3, 1+x, 1+x^2, 1+x+x^2$. The various outputs are 0000, 00011, 00001, 000111, 000011, 0000101, 000101, 000001, 0000111, 0000011, 0000001. These can be rewritten as 0000000, 0000100, 0000100, 0001110, 0000110, 0000101, 0001010, 0000010, 0000111, 0000011, 0000001. Hence K_4 can be coded as

$C=\{0000,0001,0010,0011,0100,0101,0110,0111,1010,1100,1110\}$.

Similarly, K_5 can be coded as

$C=\{00000,00001,00011,00100,00101,00110,00111,01001,01011,01100,01101,$

$01110,01111,10010,10100,10110,11000,11010,11100,11110\}$.

From the above observations can be stated as the following results.

Result 1.5 Using vertex polynomial, K_n can be coded with codewords of length *n* and cyclic deficiencies 3, 4, 12,... for $n=3,4,5,\ldots$.

Next, try to code the graph P_n using neighbourhood polynomial and vertex polynomial

(i) Using Neighbourhood Polynomial

Consider P_3. Then corresponding $m(x)=1+x+x^2$. The possible distinct $g_i(x)$ are $1, 1+x, 1+x+x^2$. The various outputs are 111, 1001, 10101. Leave the output corresponding to the null graph. Then we get the output streams are 1001 and 10101. These can be rewritten as 10010 and 10101. Hence P_3 can be coded as $C=\{00,01,10,11\}$.

Consider P_4. Then corresponding $m(x)=1+x+x^2$. The possible distinct $g_i(x)$ are $1,1+x,1+x+x^2$ The various outputs are 111, 1001, 10101. Leave the output corresponding to the null graph. Then we get the output streams are 1001 and 10101. These can be rewritten as 10010 and 10101. Hence P_4 can be coded as $C=\{00,01,10,11\}$.

Similarly, P_5 can be coded as $C=\{00,01,10,11\}$, P_6 can be coded as $C=\{00,01,10,11\}$.

Now, generalize the above results as follows:

Result 1.6 The graph $P_n, n>2$ can be coded as $C=\{00,01,10,11\}$ using neighbourhood polynomial.

(ii) Using Vertex Polynomial

Consider P_3. Then corresponding $m(x)=x+x^2$. The possible distinct $g_i(x)$ are $0,1+x,x+x^2$. The various outputs are 000, 0101, 00101. Leave the output corresponding to the null graph. Then we get the output streams are 0101 and 00101. These can be rewritten as 01010 and 10101. Hence P_3 can be coded as $C=\{00,01,10\}$.

Consider P_4. Then corresponding $m(x)=x+x^2$. The possible distinct $g_i(x)$ are $0,x,1+x,x+x^2,1+x+x^2$. The various outputs are 000, 0101, 01001, 0011, 00101. Leave the output corresponding to the null graph. Then we get the output streams are 0101, 01001, 0011 and 00101. These can be rewritten as 01010, 01001, 00110 and 00101. Hence P_4 can be coded as $C=\{0000,1100,0011,0101\}$.

Similarly, P_5 can be coded as $C=\{000,110,001,100,011\}$, P_6 can be coded as $C=\{0000,1100,0011,0101\}$, P_7 can be coded as $C=\{000,110,001,100,011\}$, P_8 can be coded as $C=\{0000,1100,0011,0101\}$.

Next, generalize the above observations as follows:

Result 1.7 The graph P_n can be coded as $C=\{0000,1100,0011,1010,0101\}$ if *n* is an odd number greater than 3 and coded as $C=\{000,110,001,100,011\}$ if *n* is an even number greater than 2. Both codes have cyclic deficiency 2.

Next, try to code the star graph $K_{1,\,n}$ using neighbourhood polynomial and vertex polynomial.

(i) Using Neighbourhood Polynomial

Consider $K_{1,\,3}$. Then corresponding $m(x)=1+x+x^2+x^3$. The possible distinct $g_i(x)$ are $1,1+x,1+x+x^2,1+x+x^2+x^3$. The various outputs are 1111, 10001, 101101, 1010101.
Leave the output corresponding to the null graph. Then we get the output streams are 10001, 101101 and 1010101. These can be rewritten as 1000100, 1011010 and 1010101. Hence $K_{1,\,3}$ can be coded as $C=\{000,001,010,011,101,111\}$.

Consider $K_{1,\,4}$. Then corresponding $m(x)=1+x+x^2+x^3+x^4$. The possible distinct $g_i(x)$ are $1,1+x,1+x+x^2,1+x+x^2+x^3,1+x+x^2+x^3+x^4$. The various outputs are 11111, 100001, 1011101, 10100101, 101010101. Leave the output corresponding to the null graph. Then we get the output streams are 100001, 1011101, 10100101 and 101010101. These can be rewritten as 100001000, 101110100, 101001010 and 101010101. Hence $K_{1,\,4}$ can be coded as $C=\{0000,0001,0010,0100,0101,0111,1010,1111\}$.

Similarly, $K_{1,\,5}$ can be coded as

$C=\{00000,00001,00010,00101,01010,01000,01111,10101,01011,11111\}$, $K_{1,\,6}$ can be coded as

$C=\{000000,000001,000010,000101,001010,010100,010101,010000,010111,011111,101010,111111\}$.

(ii) Using Vertex Polynomial

Consider $K_{1,\,3}$. Then corresponding $m(x)=x+x^3$. The possible distinct $g_i(x)$ are $0,1+x,1+x+x^2,x+x^3$. The various outputs are 0000, 01111, 011011, 0010001. Leave the output corresponding to the null graph. Then we get the output streams are 01111, 011011 and 0010001. These can be rewritten as 0111100, 0110110 and 0010001. Hence $K_{1,\,3}$ can be coded as $C=\{000,001,010,100,110,111\}$.

Consider $K_{1,\,4}$. Then corresponding $m(x)=x+x^4$. The possible distinct $g_i(x)$ are $0,1+x,1+x+x^2,1+x+x^3,x+x^4$. The various outputs are 00000, 011011, 0111111, 01100101, 001000001. Leave the output corresponding to the null graph. Then we get the output streams are 011011, 0111111, 01100101 and 001000001. These can be rewritten as 011011000, 011111100, 011001010 and 001000001.

Hence $K_{1,\,4}$ can be coded as $C=\{0000,0001,0010,0100,1100,1110,1111\}$.

Similarly, $K_{1,\,5}$ can be coded as $C=\{00000,00001,00010,00100,01000,11100,11110,11111\}$, $K_{1,\,6}$ can be coded as $C=\{000000,000001,000010,000100,001000,010000,111100,111110,111111\}$, $K_{1,\,7}$ can be coded as $C=\{0000000,0000001,0000010,0000100,0001000,0010000,0100000,1111100,\ 1111110,1111111\}$.

Now, generalize the above findings as follows:

Result 1.8 The star graph $K_{1,\,n}$ can be coded with codewords of length *n* with cyclic deficiencies 2, 4, 12, 16,... for *n* = 3, 4, 5, 6,... using neighbourhood polynomial and with cyclic deficiencies 2, 7, 9, 11, 13,... for *n* = 3, 4, 5, 6, 7,... using vertex polynomial.

Next, try to code the wheel graph W_n using neighbourhood polynomial and vertex polynomial.

(i) Using Neighbourhood Polynomial

Consider W_4. Then corresponding $m(x)=1+x+x^2+x^3$. The possible distinct $g_i(x)$ are $1,1+x,1+x+x^2,1+x+x^2+x^3$. The various outputs are 1111, 10001, 101101, 1010101. Leave the output corresponding to the null graph. Then we get the output streams are 10001, 101101 and 1010101. These can be rewritten as 1000100, 1011010 and 1010101. Hence W_4 can be coded as $C=\{000,001,010,011,101,111\}$.

Consider W_5. Then corresponding $m(x)=1+x+x^2+x^3+x^4$. The possible distinct $g_i(x)$ are $1,1+x,1+x+x^2,1+x+x^2+x^3,1+x+x^2+x^3+x^4$. The various outputs are 11111, 100001, 1011101, 10100101, 101010101. Leave the output corresponding to the null graph. Then we get the output streams are 100001, 1011101, 10100101 and 101010101. These can be rewritten as 100001000, 101110100, 101001010 and 101010101. Hence W_5 can be coded as $C=\{0000,0001,0010,0100,0101,0111,1010,1111\}$.

Similarly, W_6 can be coded as

$C=\{00000,00001,00010,00101,01010,01000,01111,10101,01011,11111\}$, W_7 can be coded as

$C=\{000000,000001,000010,000101,001010,010100,010101,010000,\ 010111,011111,101010,111111\}$,

W_8 can be coded as

$C=\{0000000,0000001,0000010,0000101,0010101,0100000,0101000,0101010,$

$0101011,0101111,011111,0001010,1010101,1111111\}$.

(ii) Using Vertex Polynomial

Consider W_4. Then corresponding $m(x)=x^3$.

The possible distinct $g_i(x)$ are $0,1+x,1+x+x^2,x,x+x^3,1+x^2,x+x^2,x+x^2+x^3,x^2,x^2+x^3,x^3$.

The various outputs are 0000, 00001, 000001, 0000001, 00011, 000101, 000011, 0000101, 0000011, 000111, 0000111. These can be rewritten as 0000000, 0000100, 0000010, 0000001, 0001100, 0001010, 0000110, 0000101, 0000011, 0001110 and 0000111. Leave the first three digits from the output streams and hence W_4 can be coded as

$C=\{0000,0001,0010,0011,0100,0101,0110,0111,1010,1100,1110\}$.

Consider W_5. Then corresponding $m(x)=x^3+x^4$. The possible distinct $g_i(x)$ are

$$0,1+x,1+x+x^2,1+x+x^3,x+x^2,1+x^2,x+x^2+x^3,1+x+x^2+x^3,x+x^4,$$

$$1+x^2+x^3,x^2,x+x^2+x^4,x^2+x^3,x+x^2+x^3+x^4,x^2+x^4,x^2+x^3+x^4,x^3+x^4.$$

The various outputs are 00000, 000101, 0001001, 00010111, 0000101, 0001111, 00001001, 00010001, 000011011, 00011101, 0000011, 000010111, 00000101, 000010001, 000001111, 000001001, 000000101.

These can be rewritten as 000000000, 000101000, 000100100, 000101110, 000010100, 000111100, 000010010, 000100010, 000011011, 000111010, 000001100, 000010111, 000001010, 000010001, 000001111, 000001001 and 000000101. Leave the first three digits from the output streams and hence W_5 can be coded as

$$C = \{000000, 000101, 001001, 001010, 001100, 001111, 010001, 010010, 010100, 010111, 011011, 100100, 101000, 101110, 100010, 111010, 111100\}.$$

Similarly, W_6 can be coded as

$$C = \{00000000,\ 01010000,\ 00101000,\ 11110000,\ 11011000,\ 11100100, 10001000, 11001100,\ 11010010,\ 01111000,\ 01000100,\ 11111010,\ 10011100,\ 01101100, 01110010,\ 01010101,\ 11000110,\ 01100110,\ 10000010,\ 01111101,\ 10010110, 00111100,\ 01101001,\ 00100010,\ 00110110,\ 00111001,\ 00010001\}.$$

Next, conclude the above results as follows:

Result 1.9 The wheel graph W_n can be coded with codewords of length $n-1$ with cyclic deficiencies 2, 4, 12, 16, 30,... for *n* = 4, 5, 6, 7, 8,... using neighbourhood polynomial and coded with cyclic deficiencies 4, 14, 32... for *n* = 4, 5, 6,... using vertex polynomial.

Next, try to code the ladder graph L_n using neighbourhood polynomial and vertex polynomial.

(i) Using Neighbourhood Polynomial

Consider L_1. Then corresponding $m(x) = 1 + x$. The possible distinct $g_i(x)$ are $1, 1+x$. The various outputs are 11, 101. Leave the output corresponding to the null graph. Then we get the output streams are 101. Hence L_1 can be coded as $C = \{0,1\}$.

Consider L_2. Then corresponding $m(x) = 1 + x + x^2$. The possible distinct $g_i(x)$ are $1, 1+x, 1+x+x^2$ The various outputs are 111, 1001, 10101. Leave the output corresponding to the null graph. Then we get the output streams are 1001 and 10101. These can be rewritten as 10010 and 10101. Hence L_2 can be coded as $C = \{00, 01, 10, 11\}$.

Similarly, L_3 can be coded as $C = \{000, 001, 010, 011, 101, 111\}$, L_4 can be coded as $C = \{000, 001, 010, 011, 101, 111\}$, L_5 can be coded as $C = \{000, 001, 010, 011, 101, 111\}$.

(ii) Using Vertex Polynomial

Consider L_2. Then corresponding $m(x) = x^2$. The possible distinct $g_i(x)$ are $0, x, x^2, 1+x, 1+x+x^2, x+x^2$. The various outputs are 000, 0001, 00001, 0011, 00011, 00111. These can be rewritten as 00000, 00010, 00001, 00110, 00011, 00111. Leave the first two digits from the output streams. Then we get the output streams are 000, 010, 001, 110, 011 and 111. Hence L_2 can be coded as $C = \{000, 001, 010, 011, 110, 111\}$.

Consider L_3. Then corresponding $m(x) = x^2 + x^3$. The possible distinct $g_i(x)$ are

$$0, x, x^2, 1+x, 1+x+x^2, 1+x+x^3, 1+x+x^2+x^3, 1+x^2, x+x^2, x+x^3, x^2+x^3, x+x^2+x^3.$$

The various outputs are 0000, 00101, 00011, 001001, 000101, 000011, 001111, 0010001, 0001001, 0000101, 0010111, 0001111. These can be rewritten as 0000000, 0010100, 0001100, 0010010, 0001010, 0000110, 0011110, 0010001, 0001001, 0000101, 0010111, 0001111.

Leave the first two digits from the output streams. Then we get the output streams are 00000, 10100, 01100, 10010, 01010, 00110, 11110, 10001, 01001, 00101, 10111 and 01111.

Hence L_3 can be coded as

$$C = \{00000, 10100, 01100, 10010, 01010, 00110, 11110, 10001, 01001, 00101, 10111, 01111\}$$

Similarly, L_4 can be coded as

$$C = \{00000, 10100, 01100, 10010, 01010, 00110, 11110, 10001, 01001, 00101, 10111, 01111, 11101\}$$

Now, conclude the above observations as follows:

Result 1.10 The ladder graph L_n can be coded as $C = \{000, 001, 010, 011, 101, 111\}$ with cyclic deficiency 2 for *n* = 3, 4, 5,... using neighbourhood polynomial. Using Vertex polynomial L_n can be coded with cyclic deficiencies 2, 4, 3,... for *n* = 2, 3, 4,....

Conclusion

Codes derived from graphs provide a powerful and flexible framework for designing efficient error-correcting codes by leveraging the properties of graphs. The interplay between graph theory and coding theory allows for the development of codes with desirable characteristics. By exploring different classes of graphs and their associated parameters, researchers can generate new families of codes with practical applications in communication, data storage, and area security. Continued research in this area promises further innovations and deeper theoretical insights, strengthening the connection between graph-based models and modern coding systems.

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