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A study on the weighted Ola distribution: Statistical properties and lifetime data applications

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Abstract
In this study, a new probability model named the Weighted Ola distribution is proposed as an extension of the original Ola distribution to improve its flexibility for modeling lifetime data. Several statistical properties of the proposed distribution are derived, including moments, order statistics, and various reliability measures. Parameter estimation is performed using the method of maximum likelihood. The applicability and efficiency of the model are demonstrated through its fit to a real lifetime dataset. Comparative results based on standard model selection criteria indicate that the Weighted Ola distribution provides a better fit compared to some existing lifetime models.

Keywords: Weighted distribution, Ola distribution, lifetime data, reliability analysis, order statistics, moment, model selection criteria

1. Introduction

Weighted distributions are widely used in fields like biomedicine, reliability, ecology, economics, and branching processes. They are helpful when not all observations have the same chance of being recorded, which leads to biased or unequal sampling. To correct this, a weight function w(x) is applied to the original probability density function f(x), creating a new distribution that more accurately represents real-life data. The idea of weighted distributions was first introduced by Fisher in (1934) [1] to deal with bias in how data is collected. Later, in (1965) [2], Rao expanded this idea to cover cases where data points don't all have the same chance of being chosen, especially in discrete distributions. Since then, many researchers have developed weighted forms of well-known distributions to make them more flexible and better suited for real-world data. In medical research, especially in cancer studies, survival analysis is very important. Researchers often look at how long it takes for an event to happen, like death or the return of cancer. This kind of time-based data is usually modeled using continuous probability distributions. But in many cases, patients who live longer are more likely to be included in the study, which creates sampling bias. Standard distributions may not handle this well. Weighted distributions help solve this problem by adjusting the probabilities to account for this unequal chance of selection. This leads to better estimates, improved model fitting, and more accurate results when studying cancer survival.

Over time, many researchers have developed weighted versions of classical probability distributions to improve flexibility and better handle real-world data issues such as biased sampling. Zakerzadeh and Dolati (2009) [4] introduced the Generalized Lindley distribution, extending the standard Lindley model. Alzaatreh *et al*. (2013) proposed the T-X family, a general method for creating new continuous distributions, including exponential-type models. Ahmad A, Ahmad SP & Ahmed A, (2016) [6] introduced the length-biased weighted Lomax distribution, a true example of a length-biased weighted model applied to real data.

Elbatal and Aryal (2017) [7] introduced the Weibull-Pareto distribution, a flexible generalization that combines the properties of the Weibull and Pareto models, to better model skewed lifetime data. Elangovan, R., & Anthony, M. (2020) [9] Demonstrates its application to real survival data and compares its performance with the original OM distribution, showing improved fit.

Helal *et al*. (2022) [13] proposed the weighted Shanker distribution; Mohiuddin *et al*. (2022) [14] introduced the weighted Amarendra distribution for medical data; and Chesneau *et al*. (2022) [15] suggested a modified weighted exponential model for skewed datasets.

Ganaie and Rajagopalan (2023) [16] developed the weighted power quasi-Lindley distribution, and Hashempour and Alizadeh (2023) [17] presented a weighted half-logistic distribution with different estimation methods. Shanker R, Ray M & Prodhani HR (2023) [18] proposed Weighted Komal distribution with properties and applications to model failure-time data from engineering, while Ranade and Rather (2023) [19] introduced the weighted Sabur distribution for lifetime data modeling Abu Thaimer and Al-Omari (2025) [25] proposed the weighted Gamma-Lindley distribution, which they applied to COVID-19 data. Soujanya and Vijayasankara (2025) [22] introduced the weighted X-Rani distribution with a focus on reliability studies. These recent models demonstrate the continued interest in improving distributional models to better reflect real-world conditions and datasets.

In this research, we propose a new two-parameter lifetime distribution, referred to as the Weighted Ola Distribution (WOD)**.** This model is developed by applying a weight function to the Ola distribution, which was introduced by Al-Ta'ani and Gharaibeh (2023) [20] as a flexible one-parameter model suitable for lifetime data analysis. The proposed WOD enhances the flexibility of the original model and demonstrates superior performance when applied to real-life datasets, including cancer survival data, compared to several existing distributions.

2. Weighted Ola distribution (WOD)

A new one parameter life distribution name as Ola distribution. the probability density function (pdf) of the Ola distribution is given by

$$f(x;\beta)=\frac{\beta^8(x^7+x^3+1)e^{-\beta x}}{\beta^7+6\beta^4+5040};\ x,\beta>0 \tag{1}$$

And cumulative distribution function of the Ola distribution is given by

$$F(x;\beta)=\frac{\beta^7x^7}{\beta^7+6\beta^4+5040};\ x,\beta>0 \tag{2}$$

A random variable x with a probability density function $f(x)$ has been examined. Let $w(x)$be a weight function that is not negative. A new probability density function should be indicated.

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))}\ ;\ x>0$$

Where, $w(x)$ is the non-negative weight function, $E[w(x)]=\int w(x)f(x)dx<\infty$.

In this paper, we have to obtain the weighted version of Ola distribution. we have considered the weight function of $w(x)=x^\alpha$ To obtain the weighted Ola distribution. the probability density function of weighted Ola distribution is given by

$$f_w(x;\beta,\alpha)=\frac{x^\alpha f(x;\beta)}{E(X^\alpha)}\ ;x>0,\beta>0,\alpha>0 \tag{3}$$

Where,

$$E(X^\alpha)=\int_0^\infty x^\alpha f\,(x;\beta)dx$$

After simplification we get

$$E(X^\alpha)=\frac{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}{\beta^\alpha(\beta^7+6\beta^4+5040)} \tag{4}$$

We can obtain the pdf of the WUD by substituting the values of equations (1) and (4) into equation (3).

$$f_w(x;\beta,\alpha)=\frac{x^\alpha\left(\frac{\beta^8(x^7+x^3+1)e^{-\beta x}}{\beta^7+6\beta^4+5040}\right)}{\left(\frac{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}{\beta^\alpha(\beta^7+6\beta^4+5040)}\right)}$$

$$f_w(x;\beta,\alpha)=\frac{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\ ;x>0,\beta>0,\alpha>0 \tag{5}$$

We shall get, The cdf for the weighted Ola distribution is given by

$$F_w(x;\beta,\alpha)=\int_0^x f_w(x;\beta,\alpha)dx$$

$$=\int_0^x\left(\frac{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)dx$$

$$=\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\int_0^x x^\alpha(x^7+x^3+1)e^{-\beta x}dx$$

$$=\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\int_0^x x^{\alpha+7}e^{-\beta x}dx+\int_0^x x^{\alpha+3}e^{-\beta x}dx+\int_0^x x^\alpha e^{-\beta x}dx$$

Put $\beta x=z, x=\frac{z}{\beta}\ dx=\frac{dz}{\beta}$

$$=\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\int_0^{\beta x}\left(\frac{z}{\beta}\right)^{\alpha+7}e^{-z}\left(\frac{dz}{\beta}\right)+\int_0^{\beta x}\left(\frac{z}{\beta}\right)^{\alpha+3}e^{-z}\left(\frac{dz}{\beta}\right)+\int_0^{\beta x}\left(\frac{z}{\beta}\right)^{\alpha}e^{-z}\left(\frac{dz}{\beta}\right)$$

After simplification we get

$$F_w(x;\beta,\alpha)=\frac{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!} \qquad (6)$$

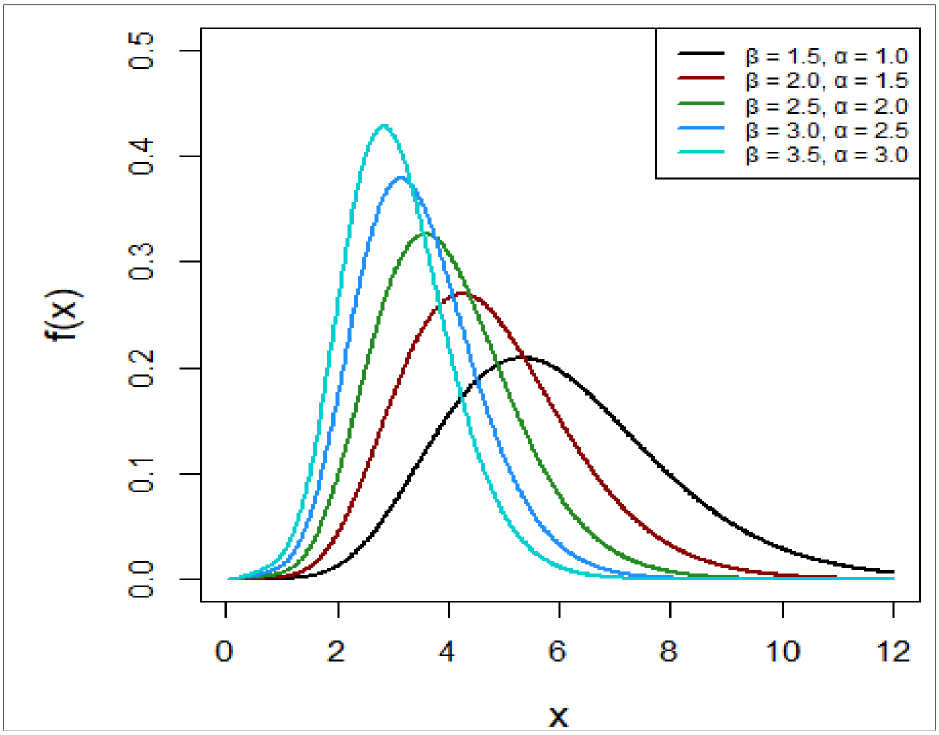

Fig 1: Pdf plot of weighted Ola distribution

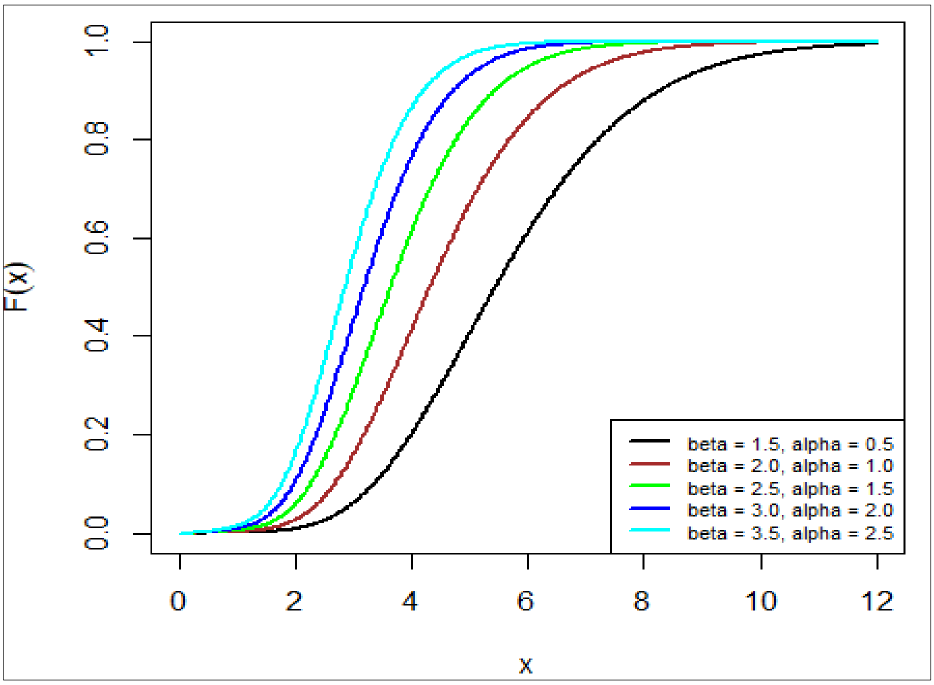

Fig 2: CDF plot of weighted Ola distribution

3. Reliability analysis

In this part, we will look at the reliability function, hazard rate, reverse hazard function, odds rate and mills ratio for the proposed weighted Ola distribution.

3.1 Reliability function

The reliability or survival function $S(x)$ is the probability of an item that will survive after a time x. using the reliability function of the weighted Ola distribution is given by

$$S(x) = 1 - F_w(x)$$

$$S(x;\beta,\alpha) = 1 - \frac{\gamma(\alpha+8,\beta x) + \beta^4\gamma(\alpha+4,\beta x) + \beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!}$$

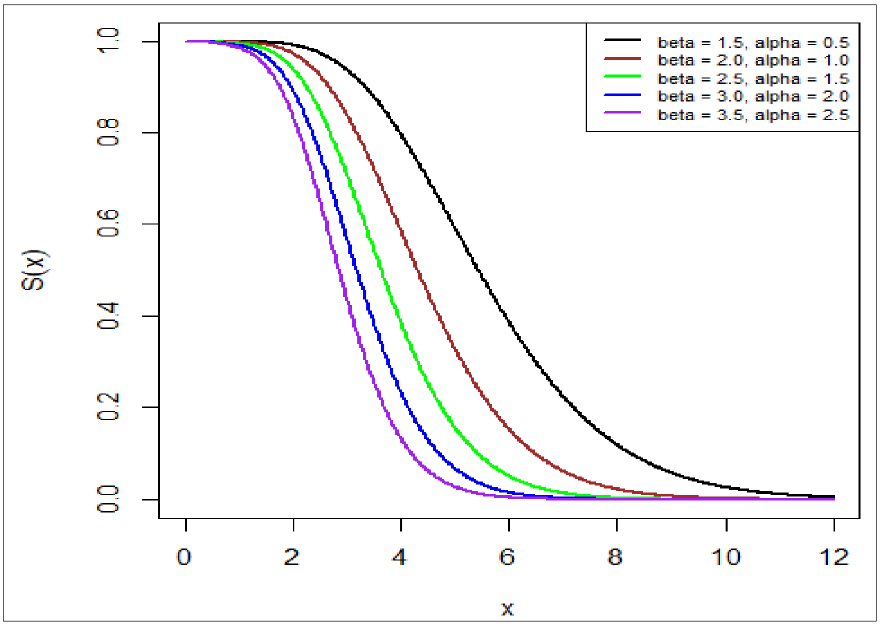

Fig 3: Survival function of weighted Ola distribution

3.2 Hazard function

The hazard function is also known as hazard rate, instantaneous failure rate or force mortality of weighted Ola distribution is given by

$$h(x;\beta,\alpha)=\frac{f_w(x;\beta,\alpha)}{S(x;\beta,\alpha)}$$

$$h(x;\beta,\alpha)=\frac{\left(\frac{x^{\alpha}\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)}{1-\left(\frac{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)}$$

$$=\frac{x^{\alpha}\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!-\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x))}$$

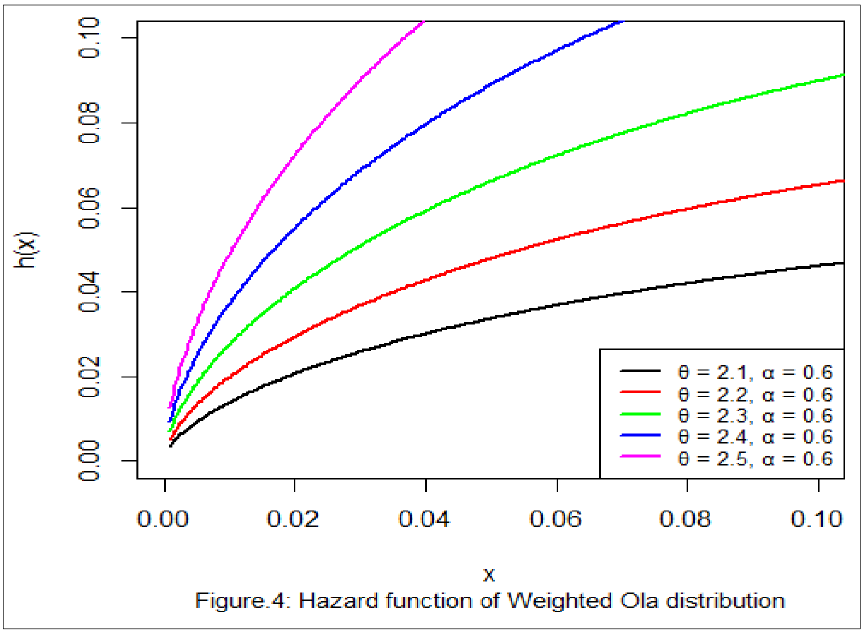

Fig 4: Hazard function of weighted Ola distribution

3.3 Revers hazard rate

Revers hazard function of weighted Ola distribution is given by

$$h_r(x;\beta,\alpha)=\frac{f_w(x;\beta,\alpha)}{F_w(x;\beta,\alpha)}$$

$$=\frac{\left(\frac{x^{\alpha}\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)}{\left(\frac{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)}$$

$$h_r(x;\beta,\alpha)=\frac{x^{\alpha}\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}$$

3.4 Odds rate function

Odds rate function of weighted Ola distribution is given by

$$O(x)=\frac{F_w(x;\beta,\alpha)}{1-F_w(x;\beta,\alpha)}$$

$$=\frac{\left(\frac{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)}{1-\left(\frac{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)}$$

$$O(x)=\frac{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!-\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}$$

3.5 Cumulative hazard function

Cumulative hazard function of weighted Ola distribution is given by

$$H(x) = -\ln(1 - F_w(x;\beta,\alpha))$$

$$H(x) = \ln\left(\frac{\gamma(\alpha+8,\beta x) + \beta^4\gamma(\alpha+4,\beta x) + \beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!} - 1\right)$$

3.6 Mills Ratio

$$Mills\ Ratio = \frac{1}{h_r(x;\beta,\alpha)}$$

$$= \frac{1}{\frac{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}}$$

$$\text{Mills Ratio} = \frac{\gamma(\alpha+8,\beta x) + \beta^4\gamma(\alpha+4,\beta x) + \beta^7\gamma(\alpha+1,\beta x)}{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}$$

4. Statistical Properties

In this section, we derived the structural properties of weighted Ola distribution

4.1 Moments

Let X_w denoted the random variable following Ola distribution then r^{th} order moment $E(X^r)$is obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f(x;\beta,\alpha)dx$$

$$\mu_r' = \int_0^\infty x^r\left(\frac{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!}\right)dx$$

$$\mu_r' = \frac{\beta^{\alpha+8}}{(\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!}\int_0^\infty x^r . x^\alpha\,(x^7+x^3+1)e^{-\beta x}dx$$

After simplification

$$E(X^r) = \mu_r' = \frac{\Gamma(r+\alpha+8) + \beta^4\Gamma(r+\alpha+4) + \beta^7\Gamma(r+\alpha+1)}{\beta^r\left((\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!\right)} \quad (7)$$

Putting r=1,2,3,4 in equation (7), the mean of weighted Ola distribution is obtained as

$$\mu_1' = \frac{\Gamma(\alpha+9) + \beta^4\Gamma(\alpha+5) + \beta^7\Gamma(\alpha+2)}{\beta\left((\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!\right)}$$

$$\mu_2' = \frac{\Gamma(\alpha+10) + \beta^4\Gamma(\alpha+6) + \beta^7\Gamma(\alpha+3)}{\beta^2\left((\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!\right)}$$

$$\mu_3' = \frac{\Gamma(\alpha+11) + \beta^4\Gamma(\alpha+7) + \beta^7\Gamma(\alpha+4)}{\beta^3\left((\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!\right)}$$

$$\mu_4' = \frac{\Gamma(\alpha+12) + \beta^4\Gamma(\alpha+7) + \beta^7\Gamma(\alpha+5)}{\beta^4\left((\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!\right)}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= \left(\frac{\Gamma(\alpha+10)+\beta^4\Gamma(\alpha+6)+\beta^7\Gamma(\alpha+3)}{\beta^2\big((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!\big)}\right) - \left(\frac{\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)}{\beta^1\big((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!\big)}\right)^2$$

$$= \frac{\left(\beta^2\big((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!\big)\right)\big(\Gamma(\alpha+10)+\beta^4\Gamma(\alpha+6)+\beta^7\Gamma(\alpha+3)\big)-\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)}{\left(\beta^2\big((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!\big)\right)^2}$$

Standard Deviation

$$S.D(\sigma) = \sqrt{\frac{\left(\beta^2\big((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!\big)\right)\big(\Gamma(\alpha+10)+\beta^4\Gamma(\alpha+6)+\beta^7\Gamma(\alpha+3)\big)-\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)}{\left(\beta^2\big((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!\big)\right)^2}}$$

4.2 Harmonic Mean

The harmonic Mean of the weighted Ola distribution is defined as

$$H.M = \int_0^\infty \frac{1}{x} f_w\,(x;\beta,\alpha)dx$$

$$H.M = \int_0^\infty \frac{1}{x}\left(\frac{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)dx$$

$$H.M = \frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\int_0^\infty \frac{1}{x}x^\alpha(x^7+x^3+1)e^{-\beta x}\,dx$$

$$H.M = \frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\int_0^\infty x^{\alpha-1}(x^7+x^3+1)e^{-\beta x}\,dx$$

$$H.M = \frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\int_0^\infty x^{\alpha+6}e^{-\beta x}\,dx + \int_0^\infty x^{\alpha+2}e^{-\beta x}\,dx + \int_0^\infty x^{\alpha-1}e^{-\beta x}\,dx$$

$$H.M = \frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\left[\frac{\Gamma(\alpha+7)}{\beta^{\alpha+7}}+\frac{\Gamma(\alpha+3)}{\beta^{\alpha+3}}+\frac{\Gamma(\alpha)}{\beta^\alpha}\right]$$

$$H.M = \frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\left[\frac{\Gamma(\alpha+7)+\beta^4\Gamma(\alpha+3)+\beta^7\Gamma(\alpha)}{\beta^{\alpha+7}}\right]$$

$$H.M = \frac{\beta^\alpha(\Gamma(\alpha+7)+\beta^4\Gamma(\alpha+3)+\beta^7\Gamma(\alpha))}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}$$

4.3 Moment Generating function and Characteristic function

Let X_w follows weighted Ola distribution then the Moment Generating Function (MGF) of X is obtained as

$$M_X(t) = E(e^{tx})$$

$$M_X(t) = \int_0^\infty e^{tx} f(x;\beta,\alpha)dx$$

Using Taylor's series expansion

$$M_X(t) = \int_0^\infty \left(1+tx+\frac{(tx)^2}{2!}+\frac{(tx)^3}{3!}+\cdots\right)f(x)dx$$

$$M_X(t) = \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!}x^j f(x)dx$$

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^{\infty} x^j f(x) dx$$

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j'$$

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j'$$

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{\Gamma(r+\alpha+8) + \beta^4\Gamma(r+\alpha+4) + \beta^7\Gamma(r+\alpha+1)}{\beta^j\left((\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!\right)}$$

$$M_X(t) = \frac{1}{(\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!} \sum_{j=0}^{\infty} \frac{t^j}{j!\,\beta^j} \left(\Gamma(r+\alpha+8) + \beta^4\Gamma(r+\alpha+4) + \beta^7\Gamma(r+\alpha+1)\right)$$

Similarly, we can get the characteristic function of weighted Ola distribution can be obtained as

$$\phi_X(t) = M_X(it)$$

$$\phi_X(t) = \sum_{j=0}^{\infty} \frac{it^j}{j!} \mu_j'$$

$$\phi_X(t) = \frac{1}{(\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!} \sum_{j=0}^{\infty} \frac{it^j}{j!\,\beta^j} \left(\Gamma(r+\alpha+8) + \beta^4\Gamma(r+\alpha+4) + \beta^7\Gamma(r+\alpha+1)\right)$$

5. Order Statistics

Let $X_{(1)}, X_{(2)}, X_{(3)}, \dots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, X_3, \dots, X_n$ drawn from the Continuous Population with probability density function $f_X(x)$ and $F_X(x)$ then the pdf of r^{th} order statistics $X_{(r)}$ is given by

$$f_{X(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!} f_X(x)[F_X(x)]^{r-1}[1 - F_X(x)]^{n-r} \quad (8)$$

Inserting equation (5) and (6) in equation (8), the probability density function of r^{th} order statistic $X_{(r)}$ of the weighted Ola distribution is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} \left(\frac{x^{\alpha}\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right) \left[\frac{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right]^{r-1} \left[1 - \frac{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right]^{n-r}$$

Therefore, the pdf of higher order statistics $X_{(n)}$ of weighted Ola distribution can be obtained as

$$f_{X_{(n)}}(x) = n \left(\frac{x^{\alpha}\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right) \left[\frac{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right]^{n-1}$$

For r = 1, we will determine the probability density function of first order statistic $X_{(1)}$ of weighted Ola distribution as

$$f_{X_{(1)}}(x) = n \left(\frac{x^{\alpha}\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right) \left[1 - \frac{\gamma(\alpha+8,\beta x)+\beta^4\gamma(\alpha+4,\beta x)+\beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right]^{n-1}$$

Quantile function

The quantile function of a distribution with cumulative distribution function $F_w(x;\beta,\alpha)$ is defined by $q = F_q(x;\beta,\alpha)$ whether 0<q<1. Thus, the quantile function of weighted Ola distribution is the real solution of the equation.

$$1 - q = 1 - \frac{\gamma(\alpha+8,\beta x) + \beta^4\gamma(\alpha+4,\beta x) + \beta^7\gamma(\alpha+1,\beta x)}{(\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!}$$

6. Likelihood Ratio Test

This section uses the weighted Ola distribution to derive the likelihood ratio test.

Consider $X_{(1)}, X_{(2)}, X_{(3)}, \dots, X_{(n)}$ as a random sample from the weighted Ola distribution.

The null and alternative hypotheses are used to test the hypothesis.

$$H_0: f(x) = f(x;\beta)\ against\ H_1: f(x) = f_w(x;\beta,\alpha)$$

The following test statistics are used to determine if a randomly selected sample of size n originates from the Ola distribution or the weighted Ola distribution.

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{f_w(x_i;\beta,\alpha)}{f(x_i;\beta)}$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{\left(\frac{x_i^{\alpha}\beta^{\alpha+8}(x_i^7+x_i^3+1)e^{-\beta x_i}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)}{\left(\frac{\beta^8(x^7+x^3+1)e^{-\beta x}}{\beta^7+6\beta^4+5040}\right)}$$

$$\Delta = \prod_{i=1}^{n}\left(\frac{x_i^{\alpha}\beta^{\alpha+8}(x_i^7+x_i^3+1)e^{-\beta x_i}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)\left(\frac{\beta^7+6\beta^4+5040}{\beta^8(x_i^7+x_i^3+1)e^{-\beta x_i}}\right)$$

$$\Delta = \prod_{i=1}^{n}\left(\frac{x_i^{\alpha}\beta^{\alpha}(\beta^7+6\beta^4+5040)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)$$

$$\Delta = \frac{L_1}{L_0} = \left(\frac{\beta^{\alpha}(\beta^7+6\beta^4+5040)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^n \prod_{i=1}^{n} x_i^{\alpha}$$

We have rejected the null hypothesis, if

$$\Delta = \frac{L_1}{L_0} = \left(\frac{\beta^{\alpha}(\beta^7+6\beta^4+5040)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^n \prod_{i=1}^{n} x_i^{\alpha} > k$$

Equivalently, we also reject null hypothesis, where

$$\Delta^* = \prod_{i=1}^{n} x_i^{\alpha} > k\left(\frac{\beta^{\alpha}(\beta^7+6\beta^4+5040)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i^{\alpha} > k^*\ where\ k^* = k\left(\frac{\beta^{\alpha}(\beta^7+6\beta^4+5040)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^n$$

for large sample size n, 2 $\log\Delta$ is distribution as chi-square variates with one degree of freedom. Thus, we rejected the null hypothesis, when the probability value is given by $p(\Delta^* > \alpha^*)$, where $\alpha^* = \prod_{i=1}^{n} x_i^{\alpha}$ is less than level of significance and $\prod_{i=1}^{n} x_i^{\alpha}$ is the observed value of the statistics Δ^*.

7. Bonferroni and Lorenz Curves and Gini Index

In this section, we have derived the Bonferroni and Lorenz curves and Gini index from the weighted Ola distribution.

The Bonferroni and Lorenz curve is a powerful tool in the analysis of distributions and has applications in many fields, such as economies, insurance, income, reliability, and medicine. The Bonferroni and Lorenz cures for a be the random variable of a unit and $f(x)$be the probability density function of x. $f(x)dx$ will be represented by the probability that a unit selected at random is defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x\ f_w(x;\beta,\alpha)dx$$

And

$$L(p) = \frac{1}{\mu}\int_0^q x\ f_w(x;\beta,\alpha)dx$$

Where,

$q = F^{-1}(p); q \epsilon [0,1]$

And $\mu = E(X)$

Thus, the Bonferroni and Lorenz curves of our distribution are, determined by

$$\mu = \frac{\Gamma(\alpha+9) + \beta^4\Gamma(\alpha+5) + \beta^7\Gamma(\alpha+2)}{\beta^1\big((\alpha+7)! + \beta^4(\alpha+3)! + \beta^7(\alpha)!\big)}$$

$$B(p) = \frac{1}{p\left(\frac{\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)}{\beta\left((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!\right)}\right)}\int_0^q x\left(\frac{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)dx$$

$$B(p) = \frac{\beta}{p\big(\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)\big)}\int_0^q x\, x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}\ dx$$

$$B(p) = \frac{\beta}{p\big(\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)\big)}.\beta^{\alpha+8}\int_0^q x^{\alpha+1}(x^7+x^3+1)e^{-\beta x}\ dx$$

$$B(p) = \frac{\beta^{\alpha+9}}{p\big(\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)\big)}\left(\int_0^q x^{\alpha+8}e^{-\beta x}\ dx + \int_0^q x^{\alpha+4}e^{-\beta x}\ dx + \int_0^q x^{\alpha+1}e^{-\beta x}\ dx\right)$$

Put $\beta x = t, x = \frac{t}{\beta}, dx = \frac{dt}{\beta}$

When $x \longrightarrow 0, t \longrightarrow 0\ and\ x \longrightarrow x, t \longrightarrow \beta q$

$$B(p) = \frac{\beta^{\alpha+9}}{p\big(\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)\big)}\left(\int_0^{\beta q}\left(\frac{t}{\beta}\right)^{\alpha+8}e^{-t}\left(\frac{dt}{\beta}\right) + \int_0^{\beta q}\left(\frac{t}{\beta}\right)^{\alpha+4}e^{-t}\left(\frac{dt}{\beta}\right) + \int_0^{\beta q}\left(\frac{t}{\beta}\right)^{\alpha+1}e^{-t}\left(\frac{dt}{\beta}\right)\right)$$

$$B(p) = \frac{\beta^{\alpha+9}}{p\big(\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)\big)}\left(\left(\frac{1}{\beta^{\alpha+9}}\right)\int_0^{\beta q} t^{\alpha+8}e^{-t}\ dt + \left(\frac{1}{\beta^{\alpha+5}}\right)\int_0^{\beta q} t^{\alpha+4}e^{-t}\ dt + \left(\frac{1}{\beta^{\alpha+2}}\right)\int_0^{\beta q} t^{\alpha+1}e^{-t}\ dt\right)$$

$$B(p) = \frac{\beta^{\alpha+9}}{p\big(\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)\big)}\left(\left(\frac{1}{\beta^{\alpha+9}}\right)\int_0^{\beta q} t^{(\alpha+9)-1}e^{-t}\ dt + \left(\frac{1}{\beta^{\alpha+5}}\right)\int_0^{\beta q} t^{(\alpha+5)-1}e^{-t}\ dt + \left(\frac{1}{\beta^{\alpha+2}}\right)\int_0^{\beta q} t^{(\alpha+2)-1}e^{-t}\ dt\right)$$

$$B(p) = \frac{\beta^{\alpha+9}}{p\big(\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)\big)}\left(\left(\frac{1}{\beta^{\alpha+9}}\right)\gamma(\alpha+9,\beta q) + \left(\frac{1}{\beta^{\alpha+5}}\right)\gamma(\alpha+5,\beta q) + \left(\frac{1}{\beta^{\alpha+2}}\right)\gamma(\alpha+2,\beta q)\right)$$

$$B(p) = \frac{\beta^{\alpha+9}}{p\big(\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)\big)} \times \frac{\gamma(\alpha+9,\beta q) + \beta^4\gamma(\alpha+5,\beta q) + \beta^7\gamma(\alpha+2,\beta q)}{\beta^{\alpha+9}}$$

$$B(p) = \frac{\gamma(\alpha+9,\beta q) + \beta^4\gamma(\alpha+5,\beta q) + \beta^7\gamma(\alpha+2,\beta q)}{p\big(\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)\big)}$$

Where,

$L(p) = pB(p)$

$$L(p) = p\left(\frac{\gamma(\alpha+9,\beta q) + \beta^4\gamma(\alpha+5,\beta q) + \beta^7\gamma(\alpha+2,\beta q)}{p\big(\Gamma(\alpha+9)+\beta^4\Gamma(\alpha+5)+\beta^7\Gamma(\alpha+2)\big)}\right)$$

$$L(p) = \left(\frac{\gamma(\alpha+9, \beta q) + \beta^4 \gamma(\alpha+5, \beta q) + \beta^7 \gamma(\alpha+2, \beta q)}{\left(\Gamma(\alpha+9) + \beta^4 \Gamma(\alpha+5) + \beta^7 \Gamma(\alpha+2)\right)}\right)$$

Gini index

The information in the Lorenz Curve is often summarized in a single measure called the Gini index (proposed in a 1912 paper by the Italian statistician Corrado Gini. It is often used as a gauge of economic inequality, measuring income distribution. The Gini index is defined as Therefore, the Gini index is for weighted Ola distribution

$$G = 1 - 2\int_0^1 L(p)dp$$

$$G = 1 - 2\int_0^1 \left(\frac{\gamma(\alpha+9, \beta q) + \beta^4 \gamma(\alpha+5, \beta q) + \beta^7 \gamma(\alpha+2, \beta q)}{\left(\Gamma(\alpha+9) + \beta^4 \Gamma(\alpha+5) + \beta^7 \Gamma(\alpha+2)\right)}\right) dp$$

$$G = 1 - 2\left(\frac{\gamma(\alpha+9, \beta q) + \beta^4 \gamma(\alpha+5, \beta q) + \beta^7 \gamma(\alpha+2, \beta q)}{\left(\Gamma(\alpha+9) + \beta^4 \Gamma(\alpha+5) + \beta^7 \Gamma(\alpha+2)\right)}\right)$$

8. Stochastic Ordering

Stochastic ordering is an important tool in finance and reliability to assess the comparative performance of the models. Let X and Y be two random variables with pdf, cdf, and reliability functions $f(x), f(y), F(x), F(y). S(x) = 1 - F(x) and\ F(y)$

1. Likelihood ratio order $(X \leq_{LR} Y) if\ \frac{f_{X_W}(x)}{f_{Y_W}(x)}$decreases in x
2. Stochastic order $(X \leq_{ST} Y)$ if $F_{X_w}(x) \geq F_{Y_w}(x)$ for all x
3. Hazard rate order$(X \leq_{HR} Y)$ if $h_{X_w}(x) \geq h_{Y_w}(x)$ for all x
4. Mean residual life order$(X \leq_{MRL} Y)$ if $MRL_{X_w}(x) \geq MRL_{Y_w}(x)$ for all x

Show that length biased Loai distribution satisfies the strongest ordering (likelihood ratio ordering) Suppose X and Y are independent random variables with probability distribution functions $f_{w_x}(x;\beta,\alpha)$ and $f_{w_y}(x;\theta,\lambda)$.If β< θ and α <λ, then.

$$\Lambda = \frac{f_{w_x}(x;\beta,\alpha)}{f_{w_y}(x;\theta,\lambda)}$$

$$\Lambda = \frac{\left(\frac{x^\alpha \beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)}{\left(\frac{x^\lambda \theta^{\lambda+8}(x^7+x^3+1)e^{-\theta x}}{(\lambda+7)!+\theta^4(\lambda+3)!+\theta^7(\lambda)!}\right)}$$

$$\Lambda = \left(\frac{x^\alpha \beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right) \times \left(\frac{(\lambda+7)!+\theta^4(\lambda+3)!+\theta^7(\lambda)!}{x^\lambda \theta^{\lambda+8}(x^7+x^3+1)e^{-\theta x}}\right)$$

$$\Lambda = \left(\frac{x^\alpha \beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{x^\lambda \theta^{\lambda+8}(x^7+x^3+1)e^{-\theta x}}\right) \times \left(\frac{(\lambda+7)!+\theta^4(\lambda+3)!+\theta^7(\lambda)!}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)$$

$$\Lambda = \frac{\beta^{\alpha+8}(\lambda+7)!+\theta^4(\lambda+3)!+\theta^7(\lambda)!}{\theta^{\lambda+8}(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!} \times \frac{x^\alpha(x^7+x^3+1)e^{-\beta x}}{x^\lambda(x^7+x^3+1)e^{-\theta x}}$$

$$\log\Lambda = \log\left(\frac{\beta^{\alpha+8}(\lambda+7)!+\theta^4(\lambda+3)!+\theta^7(\lambda)!}{\theta^{\lambda+8}(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right) + \log(x^\alpha(x^7+x^3+1)) - \log\left(x^\lambda(x^7+x^3+1)\right) - (\beta-\theta)x$$

$$\log\Lambda = \log\left(\frac{\beta^{\alpha+8}(\lambda+7)!+\theta^4(\lambda+3)!+\theta^7(\lambda)!}{\theta^{\lambda+8}(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right) + \log(x^{\alpha+7}+x^{\alpha+3}+x^\alpha) - \log\left(x^{\lambda+7}+x^{\lambda+3}+x^\lambda\right) - (\beta-\theta)x$$

Differentiate with respect to x, we get.

$$\frac{\partial \log\Lambda}{\partial x} = \frac{(\alpha+7)x^{(\alpha+6)}+(\alpha+3)x^{(\alpha+2)}+\alpha x^{(\alpha-1)}}{x^{(\alpha+7)}+x^{(\alpha+3)}+x^\alpha} - \frac{(\lambda+7)x^{(\lambda+6)}+(\lambda+3)x^{(\lambda+2)}+\lambda x^{(\lambda-1)}}{x^{(\lambda+7)}+x^{(\lambda+3)}+x^\lambda} + (\beta-\theta) = 0$$

Hence $\frac{\partial log[\Lambda]}{\partial x} < 0\ if$ β < θ and α < λ

9. Entropies

In this part, we used the weighted Ola distribution to calculate Shannon, Renyi, and Tsallis entropies.

It is commonly known that entropy and information can be used to measure uncertainty or the randomness of probability distributions. It is used in a variety of disciplines, including engineering, finance, information theory, and biomedicine. The entropy functionals for the probability distribution were developed using a variational concept of uncertainty.

9.1 Shannon Entropy

The Shannon entropy of the random variable X that defines the weighted Ola distribution

$$s_\lambda = -\int_0^\infty f(x)\log\big(f(x)\big)dx\ ;\lambda > 0, \lambda \neq 1$$

$$s_\lambda = -\int_0^\infty f(x;\beta,\alpha)\log\big(f(x;\beta,\alpha)\big)dx$$

$$s_\lambda = -\int_0^\infty \left(\frac{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)\log\left(\frac{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)dx$$

9.2 Renyi Entropy

Entropy is described as a random variable. X represents the variation of the uncertainty. Engineering, statistical mechanics, finance, information theory, biology, and economics are among the domains where it is applied. The entropy is the Renyi of order, which is defined as

$$R_\lambda = \frac{1}{1-\lambda}\log\int_0^\infty [f(x)]^\lambda dx\ ;\lambda > 0, \lambda \neq 1$$

$$R_\lambda = \frac{1}{1-\lambda}\log\int_0^\infty [f(x;\beta,\alpha)]^\lambda dx$$

$$R_\lambda = \frac{1}{1-\lambda}\log\int_0^\infty \left(\frac{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^\lambda dx$$

$$R_\lambda = \frac{1}{1-\lambda}\log\left(\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^\lambda \int_0^\infty x^{\alpha\lambda}(x^7+x^3+1)^\lambda e^{-\lambda\beta x}dx$$

Using Binomial expansion, we get

$$R_\lambda == \frac{1}{1-\lambda}\log\left(\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^\lambda \sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\int_0^\infty x^{\alpha\lambda+3i+4j}e^{-\lambda\beta x}dx$$

Using gamma function

$$R_\lambda == \frac{1}{1-\lambda}\log\left(\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^\lambda \sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\frac{\Gamma(\alpha\lambda+3i+4j+1)}{(\lambda\beta)^{\alpha\lambda+3i+4j+1}}$$

9.3 Tsallis Entropy

The Boltzmann-Gibbs (B-G) statistical properties initiated by Tsallis have received a great deal of attention. This generalization of (B-G) statistics was first proposed by introducing the mathematical expression of Tsallis entropy (Tsallis, (1988) for continuous random variables, which is defined as

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\int_0^\infty [f_w(x)]^\lambda dx\right]\ ;\lambda > 0, \lambda \neq 1$$

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\int_0^\infty [f_w(x;\ \beta,\alpha)]^\lambda dx\right]$$

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\int_0^\infty \left(\frac{x^\alpha\beta^{\alpha+8}(x^7+x^3+1)e^{-\beta x}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^\lambda dx\right]$$

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\left(\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^\lambda \int_0^\infty x^{\alpha\lambda}(x^7+x^3+1)^\lambda e^{-\lambda\beta x}dx\right]$$

Using Binomial expansion, we get

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\left(\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^\lambda \sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\int_0^\infty x^{\alpha\lambda+3i+4j}e^{-\lambda\beta x}dx\right]$$

Then using gamma function

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\left(\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)^\lambda \sum_{i=0}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}\frac{\Gamma(\alpha\lambda+3i+4j+1)}{(\lambda\beta)^{\alpha\lambda+3i+4j+1}}\right]$$

10. Estimations of parameter

This section provides the MLE and Fisher's information matrix for the Weighted Ola distribution.

MLE and Fisher's Information Matrix

Assume $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$is a random sample of size n from the weighted Ola distribution with parameter and the likelihood function, which is defined as

$$L(x;\ \theta,\alpha) = \prod_{i=1}^{n} f_w(x_i;\beta,\alpha)$$

$$L(x;\ \theta,\alpha) = \prod_{i=1}^{n}\left(\frac{x_i{}^\alpha\beta^{\alpha+8}(x_i{}^7+x_i{}^3+1)e^{-\beta x_i}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7(\alpha)!}\right)$$

$$L(x;\ \theta,\alpha) = \left(\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!}\right)\prod_{i=1}^{n} x_i{}^\alpha(x_i{}^7+x_i{}^3+1)e^{-\beta x_i}$$

The log likelihood function is given by

$$\log L = \log\left(\frac{\beta^{\alpha+8}}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!}\right)\log\prod_{i=1}^{n} x_i{}^\alpha(x_i{}^7+x_i{}^3+1)e^{-\beta x_i}$$

$$\log L = n\log(\beta^{\alpha+8}) - n\log((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!) + \sum_{i=1}^{n}\log x_i{}^\alpha(x_i{}^7+x_i{}^3+1) + \log e^{-\beta\sum_{i=1}^{n}x_i}$$

$$\log L = n\ (\alpha+8)\log(\beta) - n\log\big((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!\big) + \textstyle\sum_{i=1}^{n}\alpha\log x_i(x_i{}^7+x_i{}^3+1) - \beta\sum_{i=1}^{n}x_i \qquad (9)$$

Equation (9) with respect to parameters β and α. We obtain the normal equations as

$$\frac{\partial logL}{\partial\beta} = \frac{n(\alpha+8)}{\beta} - n\frac{(4\beta^3(\alpha+3)!+7\beta^6\alpha!)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!} - \sum_{i=1}^{n}x_i = 0\ (10)$$

$$\frac{\partial logL}{\partial\alpha} = n\log(\beta) - n\frac{\big(\Psi(\alpha+7)+\beta^4\Psi(\alpha+3)+\beta^7\Psi(\alpha)\big)}{(\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!} + \sum_{i=1}^{n}\log x_i = 0\ (11)$$

The maximum likelihood estimates of the parameters of the distribution are obtained by solving this nonlinear system of equations. Therefore, we use R and wolfram mathematics for estimating the parameters of the newly proposed distribution.

We apply asymptotic normality results to get the confidence interval. If $\hat{\lambda} = (\hat{\theta})$ represents the MLE of $\lambda = (\theta)$, we can state the following results:$\sqrt{n}\big(\hat{\lambda}-\lambda\big) \to N_2\big(0, I^{-1}(\lambda)\big)$

Where I(λ) is Fisher's information matrix. i.e.,

$$I(\lambda) = \frac{1}{n}\begin{bmatrix} E\left[\frac{\partial^2 \log L}{\partial\alpha^2}\right] & E\left[\frac{\partial^2 \log L}{\partial\alpha\partial\beta}\right] \\ E\left[\frac{\partial^2 \log L}{\partial\beta\partial\alpha}\right] & E\left[\frac{\partial^2 \log L}{\partial\beta^2}\right] \end{bmatrix}$$

$$E\left[\frac{\partial^2 \log L}{\partial\alpha^2}\right] = n\left[\frac{\left((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!\right)*\left(\Psi\prime(\alpha+7)+\beta^4\Psi\prime(\alpha+3)+\beta^7\Psi\prime(\alpha)\right)-\left(\Psi\prime(\alpha+7)+\beta^4\Psi\prime(\alpha+3)+\beta^7\Psi\prime(\alpha)\right)^2}{\left((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!\right)^2}\right]$$

$$E\left[\frac{\partial^2 \log L}{\partial\beta^2}\right] = -n\frac{n(\alpha+8)}{\beta^2} + n\left[\frac{(12\beta^2(\alpha+3)!+48\beta^5\alpha!)*\left((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!\right)-(4\beta^3(\alpha+3)!+7\beta^6\alpha!)^2}{\left((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!\right)^2}\right]$$

$$E\left[\frac{\partial^2 \log L}{\partial\alpha\partial\beta}\right] = E\left[\frac{\partial^2 \log L}{\partial\beta\partial\alpha}\right] = \frac{n}{\beta} + n\left[\frac{(4\beta^3\Psi(\alpha+3)+7\beta^6\Psi(\alpha))*\left((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!\right)-(4\beta^3(\alpha+3)!+7\beta^6\alpha!)*\left(\Psi(\alpha+7)+\beta^4\Psi(\alpha+3)+\beta^7\Psi(\alpha)\right)}{\left((\alpha+7)!+\beta^4(\alpha+3)!+\beta^7\alpha!\right)}\right]$$

Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence interval for β, α.

11. Applications

Data set 1

This application is from [3] and it is about the remission times (in months) of 128 patients suffering from bladder cancer. This data has been analyzed recently in many papers, such as [10,12]. The dataset values are as follows:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69

Data set 2

Survival time for 44 patients diagnosed by Head and Neck cancer disease from [49] and ana lyzed recently by [11] is considered. The dataset are:

12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194,195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776

To compare to the goodness of fit of the fitted distribution, the following criteria: Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Akaike Information Criteria Corrected (AICC) and -2log L.

AIC, BIC, AICC and -2log L can be evaluated by using the formula as follows.

$$AIC = 2K - 2\log L,\ BIC = k\log n - 2\log L \ and\ AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$$

Where, k=number of parameters, n sample size and -2log L is the maximized value of loglikelihood function.

Table 1; MLEs AIC, BIC, AICC, and -2log L of the fitted distribution for the given data set 1

Distribution	ML Estimates	-2 log L	AIC	BIC	AICC
Weighted Ola distribution	$\hat{\beta}$ =1.0743(0.1292) $\hat{\alpha}$ =0.0100(0.9222)	976.5193	980.5193	986.1108	980.6209
Ola distribution	$\hat{\beta}$ =0.5977(0.0193)	1180.7568	1182.7568	1185.5442	1182.7909
Exponential distribution	$\hat{\theta}$ =0.336(0.0030)	1072.8796	1074.8796	1077.6918	1074.9130
Lindley distribution	$\hat{\theta}$ =0.595(0.0038)	1124.5673	1126.5673	1129.4194	1126.5993

Table 2; MLEs AIC, BIC, AICC, and -2log L of the fitted distribution for the given data set 2

Distribution	ML Estimates	-2 log L	AIC	BIC	AICC
Weighted Ola distribution	$\hat{\beta}$ = 0.0504 (0.0113) $\hat{\alpha}$ = 0.0100 (1.7335)	621.1025	625.1025	628.5296	625.4182
Ola distribution	$\hat{\beta}$ = 0.0353 (0.0019)	798.5262	800.5262	802.3104	800.6237
Exponential distribution	$\hat{\theta}$ =0.000509 (0.000081)	795.6823	797.6823	799.3711	797.7875
Lindley distribution	$\hat{\theta}$ = 0.0010 (0.0001)	912.5921	914.5921	916.4207	914.6830

From table 1, and 2 it can be clearly observed and seen from the results that the weighted Ola distribution have the lesser AIC, BIC, AICC, -2log$L$, and values as compared to the Ola, exponential and Lindley distributions, which indicates that the weighted Ola distribution better fits than the Ola, exponential and Lindley distributions. Therefore, it can be concluded that the weighted Ola distribution provides a better fit than the other compared distributions.

12 Conclusion

In this study, we introduced the Weighted Ola distribution as an extended form of the standard Ola distribution to provide greater flexibility for modeling lifetime data. We derived several statistical properties of the proposed distribution, including moments, reliability functions, and order statistics. The parameters were estimated using the method of maximum likelihood. The practical application of the Weighted Ola distribution was demonstrated using real-life cancer survival data. The model was compared with the original Ola, Exponential, and Lindley distributions. Based on model selection criteria such as AIC, BIC, AICC, and −2LogL, the Weighted Ola distribution showed the best fit among all the models considered. Overall, the results confirm that the Weighted Ola distribution is a more accurate and flexible model for lifetime data analysis and can be effectively used in medical and reliability studies.

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