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# A new generalization of fuyi distribution and its applications to the cancer data

**O Anu and P Pandiyan**

**Abstract**
We studied a new model function named a one parameter fuyi distribution. The developed distribution is obtained by using an fuyi distribution. This distribution is a specific type of basic distribution. The area-biased distribution is compared to the original distribution. Some statistical properties of this distribution such as moments and moment-generating functions, reliability analysis, maximum likelihood function, order statistics, Renyi entropy, Tsallis entropy, Bonferroni, and Lorenz curves are derived with the partial ordering of probability distribution. A fitting of an application to a real-life bone cancer data set reveals a good fit.

**Keywords:** Area-biased, fuyi distribution, cancer data, entropy, fisher information measure, reliability analysis

## Introduction

The concept of weighted distribution was given by Fisher (1934) to model the ascertainment bias. Later, Rao developed this concept in a unified manner while the statistical data when the standard distributions were not appropriate to record these observations with equal probabilities. Fuyi distribution is a newly proposed one parametric model and discusses its various statistical properties, including its reliability, hazard rate function, mean residual life function, shape, moments, stochastic ordering, moment generating function, skewness, kurtosis, Renyi entropy, Bonferroni, Lorenz curve, and maximum likelihood estimation. The new one-parameter life-time distribution is called the area-biased fuyi distribution. The area-biased fuyi distribution has better flexibility to compare fuyi distribution.

In this paper, a new one-parameter lifetime distribution has been proposed and named the fuyi distribution. Researchers in the fields of mathematical and biological science, demography, economics, engineering, insurance, and medical sciences have developed and used single-parameter lifetime distributions to model the varying behavioural structure of univariate lifetime data. The Lindly [1], Shankar [2], Akash [3], Rama [4], Suja [5], Sujatha [6], Amarendra [7], Devya [8], Shambu [9], Aradhana [10], Akshaya [11], Pranav [12], Ishita [13], Ram Awadh [14], Prakavmy [15], and Odoma [16] distributions are examples. In this paper our objective is to introduce a distribution that gives better fitting than both exponential and Lindley distributions for modeling real-life lifetime data sets from various fields of knowledge.

### The probability density function of Fuyi distribution (PDF)

$$f_{FD}(x,\theta) = \frac{\theta}{720(\theta^7+\theta^3+1)}(\theta^{13}x^6 + 360\theta^5x^2 + 720)e^{-\theta x}dx \quad (1)$$

### The Cumulative distribution function of Fuyi distribution (CDF)

$$F_{FD}(x,\theta) = 1 - \left[1 + \frac{\theta^4x(\theta^9x^5+6\theta^8x^4+30\theta^7x^3+120\theta^6x^2+360\theta^5x+720\theta^4+720)}{720(\theta^7+\theta^3+1)}\right] \quad (2)$$

**The Area Biased Fuyi Distribution (ABFD)**
**The probability density function of the area-biased Fuyi distribution is given by**

$f_a(x) = \frac{w(x)f(x)}{E(w(x))}; x > 0,$

Where $w(x)$ be a non-negative weight function and $E(w(x)) = \int w(x)f(x)dx < \infty$.
In this paper, we will consider the area-biased version of Fuyi distribution known as area- biased Fuyi distribution (ABFD). Consequently, $w(x) = x^2$ the resulting distribution, called area-biased Fuyi distribution is given as:

$f_a(x) = \frac{x^2 f(x)}{E(x^2)}; x > 0$

Where,

$$E(x^2) = \int_0^\infty x^2 f(x;\theta)dx \quad (3)$$

$$= \int_0^\infty x^2 \frac{\theta}{720(\theta^7 + \theta^3 + 1)} (\theta^{13}x^6 + 360\theta^5 x^2 + 720)\, e^{-\theta x}\, dx$$

$$= \frac{\theta}{720(\theta^7+\theta^3+1)} \int_0^\infty x^2(\theta^{13}x^6 + 360\theta^5 x^2 + 720)e^{-\theta x}\, dx$$

$$= \frac{\theta}{720(\theta^7+\theta^3+1)} \theta^{13} \int_0^\infty x^8 e^{-\theta x}dx + 360\theta^5 \int_0^\infty x^4 e^{-\theta x}dx + 720 \int_0^\infty x^2 e^{-\theta x}dx$$

$$= \frac{\theta}{720(\theta^7+\theta^3+1)} \theta^{13} \int_0^\infty x^{9-1} e^{-\theta x}dx + 360\theta^5 \int_0^\infty x^{5-1} e^{-\theta x}dx + 720 \int_0^\infty x^{3-1} e^{-\theta x}dx$$

$$= \frac{\theta}{720(\theta^7+\theta^3+1)} \left[\frac{\theta^{13}\,\Gamma 9}{\theta^9} + \frac{360\theta^5\,\Gamma 5}{\theta^5} + \frac{720\,\Gamma 3}{\theta^3}\right]$$

$$= \frac{\theta}{720(\theta^7+\theta^3+1)} \left[\frac{\theta^{13}\,\Gamma 9 + 360\theta^9\,\Gamma 5 + 720\theta^6\,\Gamma 3}{\theta^9}\right]$$

$$= \frac{\theta^{13}\,\Gamma 9 + 360\theta^9\,\Gamma 5 + 720\theta^6\,\Gamma 3}{\theta^8[720(\theta^7+\theta^3+1)]}$$

$$E(x^2) = \frac{40320\theta^{13} + 86640\theta^9 + 1440\theta^6}{\theta^8[720(\theta^7+\theta^3+1)]} \quad (4)$$

Substitute the equations (1) and (4) in equation (3),
We will get the required probability density function of area-biased Fuyi distribution as

$$f_a(x) = \frac{x^2 f(x)}{E(x)}$$

$$f_a(x) = \frac{x^2 \frac{\theta}{720(\theta^7+\theta^3+1)}(\theta^{13}x^6 + 360\theta^5 x^2 + 720)e^{-\theta x}}{\frac{40320\theta^{13} + 86640\theta^9 + 1440\theta^6}{\theta^8[720(\theta^7+\theta^3+1)]}}$$

$$= \frac{\theta^9 x^2(\theta^{13}x^6 + 360\theta^5 x^2 + 720)e^{-\theta x}}{40320\theta^{13} + 86640\theta^9 + 1440\theta^6}$$

$$= \frac{\theta^9 x^2(\theta^{13}x^6 + 360\theta^5 x^2 + 720)e^{-\theta x}}{\theta^6(40320\theta^7 + 86640\theta^3 + 1440)}$$

$$f_a(x) = \frac{\theta^3}{720(56\theta^7 + 12\theta^3 + 2)} x^2(\theta^{13}x^6 + 360\theta^5 x^2 + 720)\, e^{-\theta x}\, dx \quad (5)$$

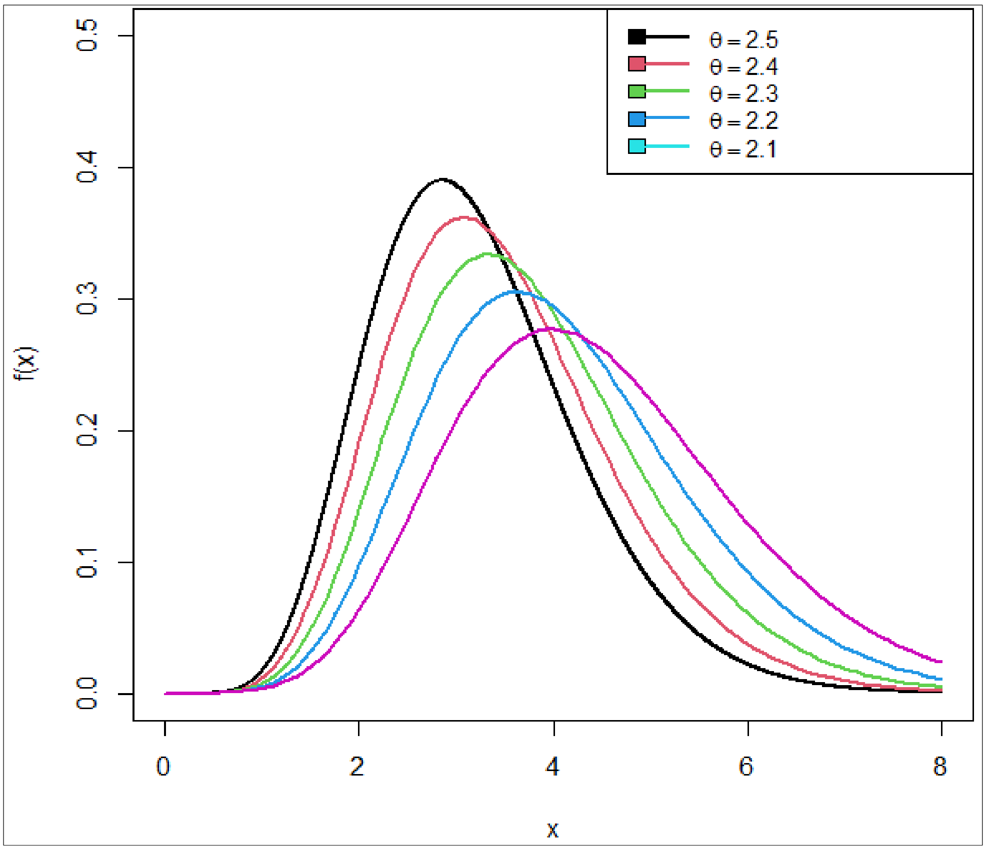


**Fig 1:** PDF plot of area-biased fuji distribution

Above the picture is Probability density function Plot of Area biased Fuyi distribution.

The cumulative distribution function (cdf) of the area-biased Fuyi distribution (ABFD).

$$F_a(x) = \int_0^x f_a(x)dx \qquad (6)$$

$$F_a(x) = \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} x^2[\theta^{13}x^6 + 360\theta^5x^2 + 720]\, e^{-\theta x}\, dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \int_0^z x^2\, [\theta^{13}x^6 + 360\theta^5x^2 + 720]\, e^{-\theta x}\, dx$$

Put $x = \frac{z}{\theta}$, $\theta x = z$, $dx = \frac{dz}{\theta}$

When $x \to 0, t \to 0$, and $x \to x, t \to \theta x$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \int_0^{\theta x} x^2[\theta^{13}x^6 + 360\theta^5x^2 + 720]\, e^{-\theta x}\, dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \theta^{13} \int_0^{\theta x} x^8\, e^{-\theta x}\, dx + 360\theta^5 \int_0^{\theta x} x^4\, e^{-\theta x}\, dx + 720 \int_0^{\theta x} x^2\, e^{-\theta x}\, dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \theta^{13} \int_0^{\theta x} \left(\frac{z}{\theta}\right)^8 e^{-z}\, \frac{dz}{\theta} + 360\theta^5 \int_0^{\theta x} \left(\frac{z}{\theta}\right)^4 e^{-z}\, \frac{dz}{\theta} + 720 \int_0^{\theta x} \left(\frac{z}{\theta}\right)^2 e^{-z}\, \frac{dz}{\theta}$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \theta^{13} \int_0^{\theta x} \frac{1}{\theta^9} z^8 e^{-z} dz + 360\theta^5 \int_0^{\theta x} \frac{1}{\theta^5} z^4 e^{-z} dz + 720 \int_0^{\theta x} \frac{1}{\theta^3} z^2 e^{-z} dz$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \frac{\theta^{13}}{\theta^9} \int_0^{\theta x} z^{9-1} e^{-z} dz + \frac{360\theta^5}{\theta^5} \int_0^{\theta x} z^{5-1} e^{-z} dz + 720 \frac{1}{\theta^3} \int_0^{\theta x} z^{3-1} e^{-z} dz$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \left(\theta^4\gamma(9,\theta x) + 360\gamma(5,\theta x) + \frac{720}{\theta^3}\gamma(3,\theta x)\right)$$

$$F_a(x) = \frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)} \qquad (7)$$

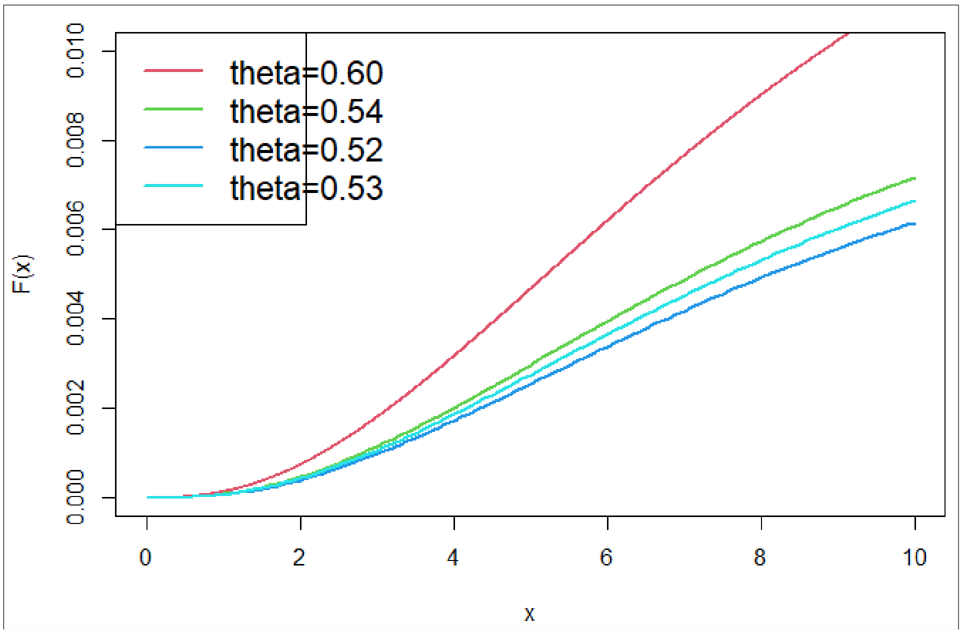


**Fig 2:** CDF plot of area-biased Fuzi distribution

Above the picture is Cumulative distribution function Plot of Area biased Fuyi distribution.

**Reliability Analysis**
We will discuss the survival function, failure rate, reverse hazard rate and the Mills ratio of the Area-biased Fuyi distribution (ABFD).

**Survival function**
The survival function of the Area-biased Fuyi distribution is given by

$$S(x) = 1 - F_a(x;\theta)$$
$$S(x) = 1 - \left(\frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)}\right) \quad (8)$$

**Hazard function**
**The hazard function is also known as the hazard rate is given by**

$$h(x) = \frac{f(x)}{1-F(x)}$$

$$h(x) = \frac{f_a(x;\theta)}{1-F_a(x;\theta)}$$

$$h(x) = \frac{\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}x^2(\theta^{13}x^6 + 360\theta^5x^2 + 720)\,e^{-\theta x}}{1 - \frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)}}$$

$$h(x) = \left(\frac{\theta^3x^2[\theta^{13}x^6+360\theta^5x^2+720]\,e^{-\theta x}}{720(56\theta^7+12\theta^3+2)-\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}\right) \quad (9)$$

**The Reverse hazard rate**
**The reverse hazard rate is given by**

$$h_r(x) = \frac{f_a(x)}{F_a(x)}$$

$$h_r(x) = \frac{f_a(x;\theta)}{F_a(x;\theta)}$$

$$h_r(x) = \frac{\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}x^2(\theta^{13}x^6+360\theta^5x^2+720)\,e^{-\theta x}}{\frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)}}$$

$$h_r(x) = \left(\frac{\theta^3x^2[\theta^{13}x^6+360\theta^5x^2+720]\,e^{-\theta x}}{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}\right) \quad (10)$$

**Odds Rate function**
**Odds Rate function of the Area biased fuyi distribution**

$$O(x) = \frac{F_a(x)}{1-F_a(x)}$$

$$O(x) = \frac{F_a(x;\theta)}{1-F_a(x;\theta)}$$

$$O(x) = \left(\frac{\frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)}}{1-\frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)}}\right)$$

$$O(x) = \left(\frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)-\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}\right) \quad (11)$$

**Cumulative hazard rate function**
The cumulative hazard function, or cumulative hazard rate, represents the total accumulated risk of experiencing an event to time t. In other words, it's a sum of (small) probabilities.

**Cumulative hazard rate function of the Area biased fuyi distribution**

$$H(x) = -\ln(1 - F_a(x)$$

$$H(x) = -\ln(1 - F_a(x;\theta)$$

$$H(x) = -\ln\left(1 - \frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)}\right)$$

**Mills Ratio**
**The Mills ratio of the area-biased Fuyi distribution is**

$$\text{Mills Ratio} = \frac{1}{h_r(x)}$$

$$\text{Mills Ratio} = \left(\frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{\theta^3x^2[\theta^{13}x^6+360\theta^5x^2+720]\,e^{-\theta x}}\right)$$

**Moments and associated measures**
Let X denote the random variable of area-biased Fuyi distribution with parameter $\theta$ then the $r^{th}$order moments $E(X^r)$of area-biased Fuyi distribution are obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r\, f_a(x)dx$$

$$= \int_0^\infty x^r \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} x^2(\theta^{13}x^6+360\theta^5x^2+720)\, e^{-\theta x}\, dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \int_0^\infty x^{r+2}\,(\theta^{13}x^6+360\theta^5x^2+720)\, e^{-\theta x}\, dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \theta^{13} \int_0^\infty x^{r+8}\, e^{-\theta x}\, dx + 360\theta^5 \int_0^\infty x^{r+4}\, e^{-\theta x}\, dx + 720 \int_0^\infty x^{r+2}\, e^{-\theta x}\, dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \theta^{13} \int_0^\infty x^{(r+9)-1}\, e^{-\theta x}\, dx + 360\theta^5 \int_0^\infty x^{(r+5)-1}\, e^{-\theta x}\, dx + 720 \int_0^\infty x^{(r+3)-1}\, e^{-\theta x}\, dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \left(\frac{\theta^{13}\,\Gamma r+9}{\theta^{r+9}} + \frac{360\theta^5\,\Gamma r+5}{\theta^{r+5}} + \frac{720\,\Gamma r+3}{\theta^{r+3}}\right)$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \left(\frac{\theta^{13}\,\Gamma r+9+360\theta^5\,\Gamma r+5+720\,\Gamma r+3}{\theta^{r+9}}\right)$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \theta^6 \left(\frac{\theta^7\,\Gamma r+9+360\theta^3\,\Gamma r+5+720\,\Gamma r+3}{\theta^{r+9}}\right)$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \left(\frac{\theta^7\,\Gamma r+9+360\theta^3\,\Gamma r+5+720\,\Gamma r+3}{\theta^{r+3}}\right)$$

$$E(X^r) = \mu_r' = \frac{\theta^7\,\Gamma r+9+360\theta^3\,\Gamma r+5+720\,\Gamma r+3}{\theta^r 720(56\theta^7+12\theta^3+2)} \quad (12)$$

**Put r = 1,2, in the equation, we will obtain the first raw moments of area-biased Fuyi distribution, which is given by**

$$E(X^1) = \mu_1' = \frac{\theta^7 \Gamma 1+9+360\theta^3 \Gamma 1+5+720 \Gamma 1+3}{\theta[720(56\theta^7+12\theta^3+2)]}$$

$$\mu_1' = \frac{\theta^7 \Gamma 10+360\theta^3 \Gamma 6+720 \Gamma 4}{\theta[720(56\theta^7+12\theta^3+2)]}$$

$$\mu_1' = \frac{\theta^7 9!+360\theta^3 5!+720\ 3!}{\theta[720(56\theta^7+12\theta^3+2)]}$$

$$E(X^2) = \mu_2' = \frac{\theta^7 \Gamma 2+9+360\theta^3 \Gamma 2+5+720 \Gamma 2+3}{\theta^2[720(56\theta^7+12\theta^3+2)]}$$

$$\mu_2' = \frac{\theta^7 \Gamma 11+360\theta^3 \Gamma 7+720 \Gamma 5}{\theta^2[720(56\theta^7+12\theta^3+2)]}$$

$$\mu_2' = \frac{\theta^7 10!+360\theta^3 6!+720\ 4!}{\theta^2[720(56\theta^7+12\theta^3+2)]}$$

Variance = $\mu_2' - (\mu_1')^2$

$$= \left(\frac{\theta^7 10!+360\theta^3 6!+720\ 4!}{\theta^2[720(56\theta^7+12\theta^3+2)]} - \left(\frac{\theta^7 9!+360\theta^3 5!+720\ 3!}{\theta[720(56\theta^7+12\theta^3+2)]}\right)^2\right)$$

$$= \left(\frac{\theta^7 10!+360\theta^3 6!+720\ 4!}{\theta^2[720(56\theta^7+12\theta^3+2)]} - \frac{(\theta^7 9!+360\theta^3 5!+720\ 3!)^2}{(\theta[720(56\theta^7+12\theta^3+2)])^2}\right) \quad (13)$$

**Harmonic mean**

**The Harmonic mean of the proposed model can be obtained as**

$$H.M = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_a(x)dx$$

$$= \int_0^\infty \frac{1}{x} \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} x^2[\theta^{13}x^6 + 360\theta^5x^2 + 720]\, e^{-\theta x}\, dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \int_0^\infty x\,(\theta^{13}x^6 + 360\theta^5x^2 + 720)e^{-\theta x}dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \int_0^\infty (\theta^{13}x^7 + 360\theta^5x^3 + 720x)e^{-\theta x}dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \theta^{13} \int_0^\infty x^7 e^{-\theta x}\, dx + 360\theta^5 \int_0^\infty x^3\, e^{-\theta x}\, dx + 720 \int_0^\infty x\, e^{-\theta x}dx$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \left(\theta^{13} \frac{\Gamma 8}{\theta^8} + 360\theta^5 \frac{\Gamma 4}{\theta^4} + 720 \frac{\Gamma 2}{\theta^2}\right)$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \left(\frac{\theta^{13}\Gamma 8+360\theta^9\,\Gamma 4+720\theta^6}{\theta^8}\right)$$

$$= \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} \left(\frac{5040\theta^{13}+2160\theta^9+720\theta^6}{\theta^8}\right)$$

$$= \frac{\theta^6(5040\theta^7+2160\theta^3+720)}{\theta^5[720(56\theta^7+12\theta^3+2)]}$$

$$= \frac{\theta(5040\theta^7+2160\theta^3+720)}{720(56\theta^7+12\theta^3+2)} \quad (14)$$

**Moment Generating Function and Characteristics Function**

**Suppose the random variable X follows Area biased Fuyi distribution with parameters $\theta$, then the MGF of X can be obtained as:**

$$M_X(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} f_a(x;\theta)dx \quad (15)$$

**Using Taylor's Series Expansion**

$M_X(t) = \int_0^\infty \left[1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right]$

$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^\infty x^j f_a(x;\theta)dx$

$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j{}'$

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \left(\frac{\theta^4 \Gamma j+9+360 \Gamma j+5+720 \Gamma j+3}{\theta^j 720(56\theta^7+12\theta^3+2)}\right) \quad (16)$$

**Similarly the Characteristics function of Area-biased Fuyi Distribution can be obtained by**

$\phi_X(t) = M_X(it)$

$$M_X(it) = \frac{1}{720(56\theta^7+12\theta^3+2)} \sum_{j=0}^{\infty} \frac{t^j}{j!\theta^j} (\theta^4 \Gamma j + 9 + 360 \Gamma j + 5 + 720 \Gamma j + 3) \quad (17)$$

**Order Statistics**

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ drawn from the continuous population with pdf $f_x(x)$ and cdf with $F_x(x)$

**The probability density function of $r^{th}$ order statistics $X_{(r)}$ is given by**

$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)[F_X(x)]^{r-1}[1 - F_X(x)]^{n-r}$

**The probability density function of $r^{th}$ order statistics $X_{(r)}$ of Area-biased Fuyi distribution is given by**

$= \frac{n!}{(r-1)!(n-r)!} \left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)} x^2(\theta^{13}x^6 + 360\theta^5x^2 + 720)\, e^{-\theta x}\, dx\right)$

$\times \left(\frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)}\right)^{r-1}$

$$\times \left(1 - \frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)}\right)^{n-r} \quad (18)$$

Therefore,

**The Probability density function of first Order Statistics $X_1$ of Area-biased Fuyi distribution is can be obtained as**

$f_{X(1)}(x) = \frac{n(n-1)!}{(1-1)!(n-1)!} \left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)} x^2(\theta^{13}x^6 + 360\theta^5x^2 + 720)\, e^{-\theta x}\, dx\right)$

$\times \left(1 - \frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)}\right)^{n-1}$

$f_{X(n)}(x) = \frac{n!}{(n-1)!(1-1)!} \left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)} x^2(\theta^{13}x^6 + 360\theta^5x^2 + 720)\, e^{-\theta x}\, dx\right)$

$\times \left(\frac{\theta^4\gamma(9,\theta x)+360\gamma(5,\theta x)+720\gamma(3,\theta x)}{720(56\theta^7+12\theta^3+2)}\right)^{n-1}$

**Likelihood Ratio Test**

Thus the likelihood-ratio test tests whether this ratio is significantly different from one, or equivalently whether its natural logarithm is significantly different from zero.

Let $X_1, X2_{,\ldots,} X_n$ be a random sample from the Area-biased Fuyi distribution. To test the hypothesis

$H_0 : f(x) = f(x;\theta)$ against $H_{1:} f(x) = f_a(x;\theta)$

In order to test whether the random sample of size n comes from the Area-biased Fuyi distribution, the following test statistics is used by

$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{f_a(x;\theta)}{f(x;\theta)}$

$$= \prod_{i=0}^{n}\left(\frac{\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}x^2(\theta^{13}x^6+360\theta^5x^2+720)\,e^{-\theta x}}{\frac{\theta}{720(\theta^7+\theta^3+1)}(\theta^{13}x^6+360\theta^5x^2+720)e^{-\theta x}}\right)$$

$$= \prod_{i=0}^{n}\left(\frac{720(\theta^7+\theta^3+1)}{\theta}\right)\times\left(\frac{\theta^3x^2}{720(56\theta^7+12\theta^3+2)}\right)$$

$$= \prod_{i=0}^{n}\frac{\theta^2 720(\theta^7+\theta^3+1)}{720(56\theta^7+12\theta^3+2)}{x_i}^2$$

$$= \left(\frac{\theta^2 720(\theta^7+\theta^3+1)}{720(56\theta^7+12\theta^3+2)}\right)^n\prod_{i=0}^{n}{x_i}^2$$

**The null hypothesis is reject**

If, $\Delta = \left(\frac{\theta^2 720(\theta^7+\theta^3+1)}{720(56\theta^7+12\theta^3+2)}\right)^n\prod_{i=0}^{n}{x_i}^2 > k$ (or)

**Equivalently,**
**We reject the null hypothesis, if**

$$\Delta^* = \prod_{i=0}^{n}{x_i}^2 > k\left(\frac{720(56\theta^7+12\theta^3+2)}{\theta^2 720(\theta^7+\theta^3+1)}\right)^n$$

$$\Delta^* = \prod_{i=0}^{n}{x_i}^2 > k^*\ where,$$

$$k^* = k\left(\frac{720(56\theta^7+12\theta^3+2)}{\theta^2 720(\theta^7+\theta^3+1)}\right)^n \qquad (19)$$

Then
$p(\Delta^* > \lambda^*)$, where, $\lambda^* = \prod_{i=0}^{n}{x_i}^2$ is less than a specified level of significance, and $\prod_{i=0}^{n}{x_i}^2$ is the observed value of $\Delta^*$

**Maximum likelihood estimate and fisher information measure**

The MLE is a method of Estimate the parameters of an assumed probability distribution, given from some observed data. The logic of maximum likelihood is both intuitive and flexible, and as such the method has become a dominant means of statistical inference.

The Fisher information is way of measuring purpose of the money of the information that an observable random variable X about an unknown parameter $\theta$ of a distribution that models X.

$$L(x) = \prod_{i=1}^{n} f_a(x)$$

$$L(x) = \prod_{i=1}^{n}\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}{x_i}^2[\theta^{13}x^6+360\theta^5x^2+720]\,e^{-\theta x}$$

$$L(x) = \frac{\theta^{3n}}{(720(56\theta^7+12\theta^3+2))^n}\prod_{i=1}^{n}{x_i}^2(\theta^{13}x^6+360\theta^5x^2+720)e^{-\theta x} \qquad (20)$$

**The log likelihood function is given by**

$$= n\,log(\theta^3) - n\log\big(720(56\theta^7+12\theta^3+2)\big) + 2\sum_{i=1}^{n}\log{x_i}^2(\theta^{13}x^6+360\theta^5x^2+720) - \theta\sum_{i=1}^{n}{x_i}^2$$

For the purpose of obtaining the confidence interval we use the asymptotic normality results. We have that if $\hat{\lambda} = (\hat{\theta})$ denotes the MLE of $\lambda = (\theta)$ We can state the results as follows

$$\sqrt{n}\,(\hat{\lambda}-\lambda) \rightarrow N_2(0, I^{-1}(\lambda))$$

**Where, $I(\lambda)$ is Fisher's Information Matrix. i.e.**

$$I(\lambda) = -\frac{1}{n}\left[E\left[\frac{\partial^2 logL}{\partial\theta^2}\right]\right]$$

Where,

$$\left[\frac{\partial^2 logL}{\partial\theta^2}\right] = E\left[\frac{\partial}{\partial\theta}\left(\frac{\partial logL}{\partial\theta}\right)\right]$$

$$\left[\frac{\partial^2 logL}{\partial\theta^2}\right] = \frac{3n}{\theta} = \theta(0) - \frac{3n(1)}{\theta^2} = \frac{-3n}{\theta^2} \qquad (21)$$

**Bonferroni and Lorenz Curves**
The Bonferroni and Lorenz curves are used in economics in order to study income, etc., but they are used in other fields like demography, insurance, medicine, and reliability. The Bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu_1'}\int_0^q x_i{}^2 f_a(x)dx$$

$$B(p) = \frac{1}{p\mu_1'}\int_0^q x_i{}^2 f_a(x;\theta)dx \text{ and}$$

$$L(p) = \frac{1}{\mu_1'}\int_0^q x_i{}^2 f_a(x;\theta)dx$$

Where, $q = F^{-1}(p);\ q\epsilon[0,1]$ and $\mu = (x)$

**Hence, the Bonferroni and Lorenz curves of our distribution are given by**

$$\mu = \frac{\theta^7 9! + 360\theta^3 5! + 720\,3!}{\theta^3[720(56\theta^7 + 12\theta^3 + 2)]}$$

$$B(p) = \frac{1}{P\left[\frac{\theta^7 9! + 360\theta^3 5! + 720\,3!}{\theta[720(56\theta^7 + 12\theta^3 + 2)]}\right]}\int_0^q \frac{\theta^3}{720(56\theta^7 + 12\theta^3 + 2)} x^2(\theta^{13}x^6 + 360\theta^5 x^2 + 720)\, e^{-\theta x}\, dx$$

$$= \frac{\theta[720(56\theta^7 + 12\theta^3 + 2)]}{P[\theta^7 9! + 360\theta^3 5! + 720\,3!]} \times \frac{\theta^3}{720(56\theta^7 + 12\theta^3 + 2)}\int_0^q x^2(\theta^{13}x^6 + 360\theta^5 x^2 + 720)e^{-\theta x}dx$$

$$= \frac{\theta^4}{P[\theta^7 9! + 360\theta^3 5! + 720\,3!]}\int_0^\infty x^2\,(\theta^{13}x^6 + 360\theta^5 x^2 + 720)e^{-\theta x}dx$$

Put, $x = \frac{t}{\theta}, \theta x = t, dx = \frac{1}{\theta}dt$

When $x \to 0, t \to 0$, and $x \to x, t \to \theta x$

$$= \frac{\theta^4}{P[\theta^7 9! + 360\theta^3 5! + 720\,3!]}\int_0^{\theta q}\left[\theta^{13}\left(\frac{t}{\theta}\right)^8 + 360\theta^5\left(\frac{t}{\theta}\right)^4 + 720\left(\frac{t}{\theta}\right)^2\right]e^{-t}\frac{1}{\theta}dt$$

$$= \frac{\theta^4}{P[\theta^7 9! + 360\theta^3 5! + 720\,3!]}\int_0^{\theta q}\left[\theta^{13}\frac{t^8}{\theta^8} + 360\theta^5\frac{t^4}{\theta^4} + 720\frac{t^2}{\theta^2}\right]e^{-t}\frac{1}{\theta}dt$$

$$= \frac{\theta^4}{P[\theta^7 9! + 360\theta^3 5! + 720\,3!]}\int_0^{\theta q}\left[\theta^5 t^8 + 360\theta\, t^4 + 720\frac{t^2}{\theta^2}\right]e^{-t}\frac{1}{\theta}dt$$

$$= \frac{\theta^4}{P[\theta^7 9! + 360\theta^3 5! + 720\,3!]}\int_0^{\theta q}\frac{1}{\theta^2}[\theta^5 t^8 + 360\theta\, t^4 + 720t^2]\,e^{-t}\frac{1}{\theta}dt$$

$$= \frac{\theta^4}{P[\theta^7 9! + 360\theta^3 5! + 720\,3!]} \times \frac{1}{\theta^2}\left[\theta^5\int_0^{\theta q} t^8 e^{-t}dt + 360\theta\int_0^{\theta q} t^4 e^{-t}dt + 720\int_0^{\theta q} t^2 e^{-t}dt\right]$$

$$= \frac{\theta^4}{P[\theta^7 9! + 360\theta^3 5! + 720\,3!]} \times \frac{1}{\theta^2}\theta^5\int_0^{\theta q} t^{9-1}e^{-t}dt + 360\theta\int_0^{\theta q} t^{5-1}e^{-t}dt + 720\int_0^{\theta q} t^{3-1}e^{-t}dt$$

$$= \frac{\theta^4}{P[\theta^7 9! + 360\theta^3 5! + 720\,3!]} \times \frac{1}{\theta^2}[\theta^5\gamma(9,\theta q) + 360\theta\gamma(5,\theta q) + 720\gamma(3,\theta q)]$$

$$B(p) = \frac{\theta^2}{p[\theta^7 9! + 360\theta^3 5! + 720\,3!]}[\theta^5\gamma(9,\theta q) + 360\theta\gamma(5,\theta q) + 720\gamma(3,\theta q)] \quad (22)$$

Where,

**Lorenz Curves**
The Lorenz curves are given by

$$L(p) = B(p)$$

$$L(p) = \frac{p[\theta^2]}{p[\theta^7 9! + 360\theta^3 5! + 720\,3!]}[\theta^5\gamma(9,\theta q) + 360\theta\gamma(5,\theta q) + 720\gamma(3,\theta q)] \quad (23)$$

**Entropies**

Entropy is most important in a different circles such as economics, probability, statistics, physics, and communication theory. It is applied to a system diversity, randomness or uncertainty.

**Shannon Entropy**

$$S_\lambda = -\int_0^\infty f_a(x) log\left(f_a(x)\right) dx$$

$$S_\lambda = -\int_0^\infty \frac{\theta^3}{720(56\theta^7+12\theta^3+2)} x^2[\theta^{13}x^6 + 360\theta^5x^2 + 720]\, e^{-\theta x}\, dx\, log\left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)} x^2[\theta^{13}x^6 + 360\theta^5x^2 + 720]\, e^{-\theta x}\, dx\right) dx \quad (24)$$

**Renyi Entropy**

Renyi entropy is given by

$$R_\lambda = \frac{1}{1-\lambda} log \int_0^\infty \left(f(x)\right)^\lambda dx\ ;\ \lambda > 0, \lambda \neq 1$$

$$R_\lambda = \frac{1}{1-\lambda} log \int_0^\infty \left(f(x;\theta)\right)^\lambda dx$$

$$R_\lambda = \frac{1}{1-\lambda} log \int_0^\infty \left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)} x^2[\theta^{13}x^6 + 360\theta^5x^2 + 720]\, e^{-\theta x}\right)^\lambda dx$$

$$R_\lambda = \frac{1}{1-\lambda} log \left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}\right)^\lambda \int_0^\infty x^{2\lambda}(\theta^{13}x^6 + 360\theta^5x^2 + 720)^\lambda\, e^{-\lambda\theta x} dx \quad (24)$$

**Using binomial expansion**

$$= \sum_{i=0}^{\lambda} \binom{\lambda}{i} (\theta^{13}x^6 + 360\theta^5x^2 + 720)^{\lambda-i}$$

$$R_\lambda = \frac{1}{1-\lambda} log\left(\left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}\right)\right)^\lambda \sum_{i=1}^{\lambda}\sum_{j=0}^{i} \binom{\lambda}{i}\binom{i}{j} (\theta)^{13(i-j)} 360\theta^{5j} \int_0^\infty x^{(2\lambda+6i-4j+1)-1} e^{-\lambda\theta x} dx$$

$$R_\lambda = \frac{1}{1-\lambda} log\left(\left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}\right)\right)^\lambda \sum_{i=1}^{\lambda}\sum_{j=0}^{i} \binom{\lambda}{i}\binom{i}{j} (\theta)^{13(i-j)} 360\theta^{5j} \frac{\Gamma 2\lambda+6i-4j+1}{(\theta)^{2\lambda+6i-4j+1}}$$

$$R_\lambda = \frac{1}{1-\lambda} log\left(\left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}\right)\right)^\lambda \sum_{i=1}^{\lambda}\sum_{j=0}^{i} \binom{\lambda}{i}\binom{i}{j} (\theta)^{13(i-j)} 360\theta^{5j} \left(\frac{1}{\theta\lambda}\right)^{2\lambda+6i-4j+1} \Gamma 2\lambda + 6i - 4j + 1 \quad (25)$$

**Tsallis Entropy**

$$T_\lambda = \frac{1}{\lambda-1}\left(1 - \int_0^\infty \left(f_a(x;\theta)\right)^\lambda dx\right) \lambda > 0, \lambda \neq 1$$

$$T_\lambda = \frac{1}{\lambda-1}\left(1 - \int_0^\infty \left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}\right) x^2(\theta^{13}x^6 + 360\theta^5x^2 + 720)\, e^{-\theta x}\right)^\lambda dx$$

$$T_\lambda = \frac{1}{\lambda-1}\left(1 - \left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}\right)^\lambda \int_0^\infty x^{2\lambda}(\theta^{13}x^6 + 360\theta^5x^2 + 720)^\lambda\, e^{-\lambda\theta x} dx\right) \quad (25)$$

**Using binomial expansion**

$$= \sum_{i=0}^{\lambda} \binom{\lambda}{i} (\theta^{13}x^6 + 360\theta^5x^2 + 720)^{\lambda-i}$$

$$T_\lambda = \frac{1}{1-\lambda}\left(1 - \left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}\right)^\lambda \sum_{i=1}^{\lambda}\sum_{j=0}^{i} \binom{\lambda}{i}\binom{i}{j} (\theta)^{13(i-j)} 360\theta^{5j} \frac{\Gamma 2\lambda+6i-4j+1}{(\theta)^{2\lambda+6i-4j+1}}\right)$$

$$T_\lambda = \frac{1}{1-\lambda}\left(1 - \left(\frac{\theta^3}{720(56\theta^7+12\theta^3+2)}\right)^\lambda \sum_{i=1}^{\lambda}\sum_{j=0}^{i} \binom{\lambda}{i}\binom{i}{j} (\theta)^{13(i-j)} 360\theta^{5j} \left(\frac{1}{\theta\lambda}\right)^{2\lambda+6i-4j+1} \Gamma 2\lambda + 6i - 4j + 1\right) \quad (26)$$

**Data Analysis**

The data under consideration are we demonstrate the applicability of the lifetime's data relating to show that area-biased Fuyi distribution can be better than Fuyi distribution.

Consider a data represents the survival time (in days) of 64 patients with bone cancer, as follows,
0.99, 1.28, 1.77, 1.97, 2.17, 2.63, 2.66, 2.76, 2.79, 2.86, 2.99, 3.06, 3.15, 3.45, 3.71, 3.75, 3.81, 4.11, 4.27, 4.34, 4.40, 4.63, 4.73, 4.93, 4.93, 5.03, 5.16, 5.17, 5.49, 5.68, 5.72, 5.85, 5.98, 8.15, 8.62, 8.48, 8.61, 9.46, 9.53, 10.05, 10.15, 10.94, 10.94, 11.24, 11.63, 12.26, 12.65, 12.78, 13.18, 13.47, 13.96, 14.88, 15.05, 15.31, 16.13, 16.46, 17.45, 17.61, 18.20, 18.37, 19.06, 20.70, 22.54, 23.36
In order to compare the performance of area-biased Fuyi distribution with Fuyi distribution. We are using the criteria values, like $AIC$, $AICC$ and $BIC$. The better distribution corresponds to lesser values of $AIC, AICC, BIC$ and $-2\log L$ can be evaluated by using the formulas as follows:

$$AIC = 2K - 2\log L \quad AICC = AIC + \frac{2k(k+1)}{(n-k-1)} \text{ and } BIC = k\log n - 2\log L$$

Where, $K$ $is$ number of parameters, n is sample size and $-2\log L$ is the maximized value of loglikelihood function.

**Table 1:** $MLEs$ $AIC, BIC, AICC,$ and $-2\log L$ of the fitted distribution for given data set

| Distribution | ML Estimates | -2logL | AIC | BIC | AICC |
|---|---|---|---|---|---|
| Area-biased Fuyi | $\hat{\theta} = 0.60255164 + (0.05820878)$ | 388.4582 | 390.4582 | 392.6171 | 390.0645 |
| Fuyi | $\hat{\theta} = 0.862009340 + (0.04544916)$ | 518.7693 | 520.7693 | 522.9282 | 520.8335 |

From the table, it can be observed that the result is an area-biased Fuyi distribution have AIC, BIC, AICC, -2logL, and compared to the values of Fuyi distributions. Our conclusion is the area-biased Fuyi distribution, given the better fits over the above Fuyi distributions.

## Conclusion

The present study proposes a new one-parameter fuyi distribution and proposes various statistical properties of the distribution derived as probability density function, cumulative distribution function, survival, hazard, and moment-generating function. The data can be used to describe the behavioral structure of data in life from engineering and medical science. The area-biased fuyi distribution in applications has been compared for goodness of fit to fuyi distribution. The results are compared with a fuyi distribution. The area-biased fuyi distribution provides better performance than the fuyi distribution.

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