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Study on Ricci curvature tensor under conformal transformation

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Abstract
This study explores the behavior of the Ricci tensor under conformal transformations of Riemannian manifolds. Conformal transformations, which preserve angles but not lengths, play a pivotal role in differential geometry and theoretical physics, particularly in the context of conformal geometry and general relativity. These analyze how the Ricci tensor transforms when the metric tensor undergoes a conformal transformation. The analysis of the Ricci tensor to conformal transformations studies how curvature responds to the resizing of the metric tensor and is of fundamental importance to the study of geometric structures and physical phenomena in general relativity, cosmology, and conformal field theory.

Keywords: Ricci curvature tensor, scalar curvature tensor, metric tensor christoffel symbols, manifold, conformal transformation, space-time

1. Introduction

A tensor, in differential geometry [1], is a mathematical entity that represents physical or geometric quantities in a manner independent of the choice of coordinates. Under coordinate changes, its components change following specific rules and make sure that equations hold in any system. A transformation is a process that changes the coordinates, size, shape, position, or orientation of geometric objects. Exciting is the conformal transformation [2], in which the metric tensor is rescaled by a positive function e^{2f}, phrased as $\tilde{g}_{ij} = e^{2f} g_{ij}$, [5] that saves angles but rescales lengths and areas. Curvature is the property of an intrinsic deviation of space away from the locally flat model, quantified by curvature tensors, which measure the rate of change of vectors in parallel translation and the rate of fan-out and convergence of geodesics. Ricci curvature tensor R_{ij}, a contraction of the [3, 7] Riemannian curvature tensor, encodes intrinsic curvature properties of a manifold because it explains how volumes in curved space and flat space are different.

One of the most important persons to come up with the Ricci tensor is Gregorio Ricci-Curbastro, an Italian mathematician whose effort in the late 19th century created the basis of tensor calculus and curvature theory. Ricci-Curbastro and Tullio Levi-Civita developed the concept of the Ricci tensor as part of their work on absolute differential calculus (now known as tensor calculus). In 1887-1896 [10], Ricci-Curbastro introduced the Ricci tensor by contracting the Riemannian curvature tensor, defining it as $R_{ij} = R^{k}_{ikj}$. This tensor, named after him, succinctly captured the manifold's curvature properties and became a cornerstone of Einstein's general relativity, fomulated in 1915 [10].

Ricci-Curbastro's contribution provided the mathematical framework to study curvature transformations, including those under conformal changes [4, 8], enabling subsequent analyses of how curvature adapts to metric rescaling. The study of the Ricci tensor under conformal transformations is vital for applications in General relativity (e.g., analyzing asymptotically flat space-time's), geometric analysis (e.g., the Yamabe problem), and theoretical physics (e.g., conformal invariance in string theory).

Ricci-Curbastro's pioneering work on the Ricci tensor provided the essential tools to explore these transformations, bridging geometry and physics [6].

2. Preliminaries

Let (M, g) be a smooth n-dimensional Riemannian manifold, where g_{ij} is the metric tensor, $i, j = 1, \dots, n$, and $n = \dim(M)$. The indices are raised and lowered using the metric g_{ij} and its inverse g^{ij}, satisfying $g_{ik}g^{kj} = \delta^i_j$. We denote partial derivatives by $\partial_i = \partial/\partial x^i$ and covariant derivatives by ∇_i.

2.1 Conformal Transformation

A conformal transformation rescales the metric tensor g_{ij} by a positive smooth function e^{2f}. The transformed metric $\tilde{g}_{ij}$ and its inverse $\tilde{g}^{ij}$ are defined as follows:

$$\tilde{g}_{ij} = e^{2f} g_{ij} \qquad (2.1.1)$$

And

$$\tilde{g}^{ij} = e^{-2f} g^{ij}. \qquad (2.1.2)$$

2.2 Christoffel Symbols: The Christoffel symbols (connection coefficients) for the metric g_{ij} are defined by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \qquad (2.2.1)$$

Under the conformal transformation $\tilde{g}_{ij} = e^{2f} g_{ij}$, the transformed Christoffel symbols Γ^k_{ij} are:

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \nabla_j f + \delta^k_j \nabla_i f - g_{ij} g^{km} \nabla_m f \qquad (2.2.2)$$

2.3 Ricci Tensor: The Ricci tensor R_{ij} is derived from the Riemannian curvature tensor R^k_{ikj}, defined as:

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{km} \Gamma^m_{ij} - \Gamma^k_{jm} \Gamma^m_{ik}. \qquad (2.3.1)$$

The Ricci tensor is the contraction $R_{ij} = R^k_{ikj}$. For the transformed metric $\tilde{g}_{ij}$, the transformed Ricci tensor is:

$$\tilde{R}_{ij} = \partial_k \tilde{\Gamma}^k_{ij} - \partial_j \tilde{\Gamma}^k_{ik} + \tilde{\Gamma}^k_{km} \tilde{\Gamma}^m_{ij} - \tilde{\Gamma}^k_{jm} \tilde{\Gamma}^m_{ik} \qquad (2.3.2)$$

2.4 Gradient and Laplacian: The gradient of the scalar function f is:

$$\nabla_i f = \partial_i f.$$

The Laplacian of function f is the trace of the Hessian, computed using the inverse metric:

$$\nabla_k \nabla^k f = g^{km} \nabla_k \nabla_m f$$

2.5 Dimensional Parameter: The dimension of the manifold is denoted $n = dim(M)$, which appears in the transformation equation of the Ricci tensor and affects the scaling of curvature terms.

3. Confromal Transformation of Ricci Tensor

Definition 3.1 (Behavior of the Ricci Tensor under Conformal Transformation). Under the conformal transformation $\tilde{g}_{ij} = e^{2f} g_{ij}$ the transformed Ricci tensor $\tilde{R}_{ij}$ is derived from the transformed Riemannian tensor, incorporating contributions from the original curvature, the conformal factor f, and its derivatives.

Theorem 3.1. Let (M, g) be a Riemannian manifold of dimension n with Ricci tensor R_{ij}. Under the conformal transformation $\tilde{g}_{ij} = e^{2f} g_{ij,}$ the Ricci tensor transforms as follows:

$$\tilde{R}_{ij} = R_{ij} + \nabla_i \nabla_j f - (n-1) \nabla_j \nabla_i f - g_{ij} \nabla_k \nabla^k f + [(2n-3) \nabla_i f \nabla_j f - (n-2) g_{ij} \nabla_k f \nabla^k f + C \qquad (3.1.1)$$

Where,

C are terms involving the Christoffel symbols expressed as follows:

$$C = (n-2)\Gamma^m_{ij} \nabla_m f + g_{ik} \Gamma^k_{jm} \nabla^m f + g_{jm} \Gamma^m_{ik} \nabla^k f - g_{ij} \Gamma^k_{km} \nabla^m f. \qquad (3.i)$$

And

- $n = dim(M)$ (3.ii)

- Original Ricci tensor: R_{ij}. (3.iii)
- Symmetric second derivative: $\nabla_i \nabla_j f$ (3.iv)
- Asymmetric second derivative correction: $-(n-1)\, \nabla_j \nabla_i f$. (3.v)
- Laplacian term: $g_{ij} \nabla_k \nabla^k f$ (3.vi)
- Gradient directional effect: $[(2n-3)\, \nabla_i f \nabla_j f$. (3.vii)
- Gradient magnitude correction: $(n-2)\, g_{ij} \nabla_k f \nabla^k f$ (3.viii)
- Christoffel-gradient interaction: $(n-2)\, \Gamma_{ij}^{m} \nabla_m f$. (3.ix)
- Mixed Christoffel term: $g_{ik} \Gamma_{jm}^{k} \nabla^m f$. (3.x)
- Symmetric counterpart: $g_{jm} \Gamma_{ik}^{m} \nabla^k f$. (3.xi)
- Trace-related Christoffel term: $-g_{ij} \Gamma_{km}^{k} \nabla^m f$. (3.xii)

NB: Index Analysis: These indices in the transformation equation are defined as follows:
• **Indices $\boldsymbol{i}, \boldsymbol{j}$:** Represent local coordinates on the manifold M, ranging from $1\ to\ n$, where

$n = dim(M)$.

• **Index $\boldsymbol{k}$**: A summation index under the Einstein convention, running over all dimensions'

$k = 1, \ldots\ldots, n$.

• **Index** m: Appears in terms involving Christoffel symbols and the gradient $\nabla_m f$, also summing over $m = 1, \ldots, n$.

3.2 Proof of theorem 3.1

Step 1: Metric Tensor under Conformal Transformation

Consider a Riemannian manifold (M, g) of dimension n, with metric tensor g_{ij}. Under a conformal transformation defined by a positive smooth function e^{2f} where f is a conformal factor, the transformed metric and its inverse are given by (2.1.1) and (2.1.2) respectively.

Step 2: Christoffel Symbols under Conformal Transformation. The Christoffel symbols for the Levi-Civita connection of the original metric $g_{ij}$are:

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right). \quad (3.2.1)$$

For the transformed metric $\tilde{g}_{ij}$, the transformed Christoffel symbols are:

$$\tilde{\Gamma}_{ij}^{k} = \frac{1}{2} \tilde{g}^{kl} \left(\partial_i \tilde{g}_{jl} + \partial_j \tilde{g}_{il} - \partial_l \tilde{g}_{ij}\right). \quad (3.2.2)$$

Substituting $\tilde{g}_{ij} = e^{2f} g_{ij}$ and computing the partial derivatives yields:

$$\partial_i \tilde{g}_{jl} = \partial_i \left(e^{2f} g_{jl}\right) = 2e^{2f} (\partial_i f)\, g_{jl} + e^{2f} \partial_i g_{jl} \quad (3.2.3)$$

Similarly, computing $\partial_j \tilde{g}_{il}$ and $\partial_l \tilde{g}_{ij}$ gives:

$$\partial_j \tilde{g}_{jl} = \partial_j \left(e^{2f} g_{il}\right) = 2e^{2f} (\partial_j f)\, g_{il} + e^{2f} \partial_j g_{il} \quad (3.2.4)$$

And

$$\partial_l \tilde{g}_{jl} = \partial_l \left(e^{2f} g_{ji}\right) = 2e^{2f} (\partial_l f)\, g_{ji} + e^{2f} \partial_l g_{ji} \quad (3.2.5)$$

Again since $\tilde{g}^{kl} = e^{-2f} g^{kl}$, the transformed Christoffel symbols become:

$$\tilde{\Gamma}_{ij}^{k} = \frac{1}{2} \tilde{g}^{kl} \left(\partial_i \tilde{g}_{jl} + \partial_j \tilde{g}_{il} - \partial_l \tilde{g}_{ij}\right).$$

$$\tilde{\Gamma}_{ij}^{k} = \frac{1}{2}\ e^{-2f} g^{kl} \left(\left[2e^{2f} (\partial_i f) g_{ji} + e^{2f} \partial_i g_{jl}\right] + [2e^{2f} (\partial_j f)\, g_{il} + e^{2f} \partial_j g_{il]} - \right.$$

$$\left. [2e^{2f} (\partial_l f)\, g_{ji} + e^{2f} \partial_l g_{ji}\,] \right).$$

$$\tilde{\Gamma}_{ij}^{k} = \frac{1}{2}\, g^{kl} \left(\left[2(\partial_i f) g_{jl} + \partial_i g_{jl}\right] + [2(\partial_j f)\, g_{il} + \partial_j g_{il]} - [2(\partial_l f)\, g_{ji} + \partial_l g_{ji}\,] \right).$$

$\tilde{\Gamma}_{ij}^{k} = \frac{1}{2}\, g^{kl}\, (\left[2(\partial_i f) g_{ji} + 2\left(\partial_j f\right) g_{il} - 2(\partial_l f)\, g_{ji}\right] + \left[\partial_i g_{jl} + \partial_j g_{il]} - \partial_l g_{ji}\right]).$

$$\tilde{\Gamma}_{ij}^{k} = \frac{1}{2}\, g^{kl}\, [\partial_i g_{jl} + \partial_j g_{il]} - \partial_l g_{ji}] + g^{kl} \left[(\partial_i f) g_{ji} + \left(\partial_j f\right) g_{il} - (\partial_l f)\, g_{ji}\right].$$

$$\tilde{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k} + \left[(\partial_i f) g^{kl} g_{ji} + \left(\partial_j f\right) g^{kl} g_{il} - (\partial_l f) g^{kl}\, g_{ji}\right].$$

$$\tilde{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k} + \left[\delta_j^k (\partial_i f) + \delta_i^k \left(\partial_j f\right) - (\partial_l f) g^{kl}\, g_{ji}\right]$$

$$\tilde{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k} + \left[\delta_j^k \nabla_i f + \delta_i^k \nabla_j f - g^{kl}\, g_{ji} \nabla_l f\right]$$

Where $\boldsymbol{l}$ is dummy index which can take any index like say $\boldsymbol{m}$, hence the above equation is equivalent to:

$$\tilde{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k} + \delta_i^k \nabla_j f + \delta_j^k \nabla_i f - g_{ij} g^{km} \nabla_m f \qquad (3.2.6)$$

Where $\nabla_i f = \partial_i f$ is the gradient of the scalar function f.

3.3 Step 3: Ricci Tensor under Conformal Transformation.
The Ricci tensor for the original metric is defined as the contraction of the Riemann curvature tensor:-

$$R_{ij} = \partial_k \Gamma_{ij}^{k} - \partial_j \Gamma_{ik}^{k} + \Gamma_{km}^{k} \Gamma_{ij}^{m} - \Gamma_{jm}^{k} \Gamma_{ik}^{m} \qquad (3.3.1)$$

For the transformed metric, the Ricci tensor is:

$$\tilde{R}_{ij} = \partial_k \tilde{\Gamma}_{ij}^{k} - \partial_j \tilde{\Gamma}_{ik}^{k} + \tilde{\Gamma}_{km}^{k} \tilde{\Gamma}_{ij}^{m} - \tilde{\Gamma}_{jm}^{k} \tilde{\Gamma}_{ik}^{m}. \qquad (3.3.2)$$

3.4 Step 4: Evaluating $\tilde{R}_{ij}$ Step by Step
First Term: $\partial_k \tilde{\Gamma}_{ij}^{k}$Substitute the transformed Christoffel symbols:

$$\tilde{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k} + \delta_i^k \nabla_j f + \delta_j^k \nabla_i f - g_{ij} g^{km} \nabla_m f \qquad (3.4.1)$$

Compute the divergence:

$$\partial_k \tilde{\Gamma}_{ij}^{k} = \partial_k\, (\Gamma_{ij}^{k} + \delta_i^k \nabla_j f + \delta_j^k \nabla_i f - g_{ij} g^{km} \nabla_m f\,). \qquad (3.4.2)$$

$$\partial_k \tilde{\Gamma}_{ij}^{k} = \partial_k \Gamma_{ij}^{k} + \partial_k\, (\delta_i^k \nabla_j f) + \partial_k\, (\delta_j^k \nabla_i f) - \partial_k\, (g_{ij} g^{km} \nabla_m f). \qquad (3.4.3)$$

Break into components:

$\partial_k\, (\delta_i^k \nabla_j f) = \nabla_i \nabla_j f,$

$\partial_k\, (\delta_j^k \nabla_i f) = \nabla_j \nabla_i f,$

$\partial_k\, (g_{ij} g^{km} \nabla_m f) = -g_{ij} \nabla_k \nabla^k f,$

$$\partial_k \tilde{\Gamma}_{ij}^{k} = \partial_k \Gamma_{ij}^{k} + \nabla_i \nabla_j f + \nabla_j \nabla_i f - g_{ij} \nabla_k \nabla^k f. \qquad (3.4.4)$$

II. **Second Term:** $\partial_j \tilde{\Gamma}_{ik}^{k}$ **Compute:**

$\tilde{\Gamma}_{ik}^{k} = \Gamma_{ik}^{k} + \delta_i^k \nabla_k f + \delta_k^k \nabla_i f - g_{ik} g^{km} \nabla_m f.$ (3.4.5)

Since $\boldsymbol{\delta_k^k = n}$ and $\boldsymbol{g_{ik} g^{km} = \delta_i^m}$ simplify:

$\tilde{\Gamma}_{ik}^{k} = \Gamma_{ik}^{k} + \nabla_i \mathrm{f} + \mathrm{n} \nabla_i \mathrm{f} - \nabla_i \mathrm{f}$
(3.4.6)

$$\tilde{\Gamma}_{ik}^{k} = \Gamma_{ik}^{k} + \mathrm{n} \nabla_i \mathrm{f}. \qquad (3.4.7)$$

Then

$$\partial_j \tilde{\Gamma}_{ik}^{k} = \partial_j \left(\Gamma_{ik}^{k} + \mathrm{n}\nabla_i \mathrm{f}\right) \quad (3.4.8)$$

$$\partial_j \tilde{\Gamma}_{ik}^{k} = \partial_j \Gamma_{ik}^{k} + \mathrm{n}\nabla_j \nabla_i f. \quad (3.4.9)$$

3.5. Step 5: Combining the Terms Subtract the second term from the first:

$$\partial_k \tilde{\Gamma}_{ij}^{k} - \partial_j \tilde{\Gamma}_{ik}^{k} = \partial_k \Gamma_{ij}^{k} - \partial_j \Gamma_{ik}^{k} + \nabla_i \nabla_j f + \nabla_j \nabla_i f - g_{ij} \nabla_k \nabla^k f - \mathrm{n}\nabla_j \nabla_i f. \quad (3.5.1)$$

$$\nabla_j \nabla_i f - \mathrm{n}\nabla_j \nabla_i f = (1-\mathrm{n})\ \mathrm{n}\nabla_j \nabla_i f. \quad (3.5.2)$$

Thus:

$$\partial_k \tilde{\Gamma}_{ij}^{k} - \partial_j \tilde{\Gamma}_{ik}^{k} = \partial_k \Gamma_{ij}^{k} - \partial_j \Gamma_{ik}^{k} + \nabla_i \nabla_j f - (n - 1)\nabla_j \nabla_i f - g_{ij} \nabla_k \nabla^k f. \quad (3.5.3)$$

However, the full expression for $\mathbf{R}_{ij}$ requires the nonlinear terms:

$$\tilde{\Gamma}_{km}^{k} \tilde{\Gamma}_{ij}^{m} - \tilde{\Gamma}_{jm}^{k} \tilde{\Gamma}_{ik}^{m}.$$

4.1 Step 1: Substitute Christoffel Symbols

Consider a Riemannian manifold (M, g) of dimension n, with metric g_{ij}. Under the conformal transformation, the transformed metric is given by equations (2.1.1) and (2.1.2).

Reacall from equation (3.2.6), we have:

From the transformed metric, the Christoffel symbols are:

$$\tilde{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k} + \delta_i^k \nabla_j f + \delta_j^k \nabla_i f - g_{ij} \nabla^k f$$

Where $\nabla_i f = \partial_i f$, and $\nabla^k f = g^{km} \nabla_m f$.

4.2 Step 2: Expanding $\boldsymbol{\Gamma_{km}^{k} \Gamma_{ij}^{m}}$. We first evaluate the nonlinear term $\tilde{\Gamma}_{km}^{k}$ and $\tilde{\Gamma}_{ij}^{m}$:

$$\tilde{\Gamma}_{km}^{k} = \Gamma_{km}^{k} + \delta_k^k \nabla_m f + \delta_m^k \nabla_k f - g_{km} \nabla^k f \quad (4.2.1)$$

Since $\boldsymbol{\delta_k^k = n}$ we have

$$\tilde{\Gamma}_{km}^{k} = \Gamma_{km}^{k} + n\nabla_m f + \nabla_m f - g_{km} \nabla^k f \quad (4.7.2)$$

His simplifies to

$$\tilde{\Gamma}_{km}^{k} = \Gamma_{km}^{k} + n\nabla_m f + \nabla_m f - \nabla_m f \quad (4.2.2)$$

$$\tilde{\Gamma}_{km}^{k} = \Gamma_{km}^{k} + n\nabla_m f \quad (4.2.3)$$

Similarly, expanding $\boldsymbol{\Gamma_{ij}^{m}}$

$$\tilde{\Gamma}_{ij}^{m} = \Gamma_{ij}^{m} + \delta_i^m \nabla_j f + \delta_j^m \nabla_i f - g_{ij} \nabla^m f \quad (4.2.4)$$

Compute the product:

$$\tilde{\Gamma}_{km}^{k} \tilde{\Gamma}_{ij}^{m} = [\ \Gamma_{km}^{k} + n\nabla_m f\ ]\ [\Gamma_{ij}^{m} + \delta_i^m \nabla_j f + \delta_j^m \nabla_i f - g_{ij} \nabla^m f\ ]. \quad (4.2.5)$$

Expand:

$$= [\Gamma_{km}^{k} \Gamma_{ij}^{m} + \Gamma_{ki}^{k} \nabla_j f + \Gamma_{kj}^{k} \nabla_i \mathrm{f} - g_{ij} \Gamma_{km}^{k} \nabla^m f] + [n\Gamma_{ij}^{m} \nabla_m f + n\nabla_i f \nabla_j f + n\nabla_j f \nabla_i f - n g_{ij} \nabla_m f \nabla^m f.] \quad (4.2.6)$$

Simplify:

$$\tilde{\Gamma}_{km}^{k} \tilde{\Gamma}_{ij}^{m} = [\ \Gamma_{km}^{k} \Gamma_{ij}^{m} + \Gamma_{ki}^{k} \nabla_j f] + [\ \Gamma_{kj}^{k} \nabla_i \mathrm{f} - g_{ij} \Gamma_{km}^{k} \nabla^m f] + [n\Gamma_{ij}^{m} \nabla_m f + 2n\nabla_i f \nabla_j f - n g_{ij} \nabla_m f \nabla^m f]. \quad (4.2.7)$$

4.3 Step 3: Expanding $\Gamma^{k}_{jm}\Gamma^{m}_{ik}$, Computing we get:

$$\tilde{\Gamma}^{k}_{jm} = \Gamma^{k}_{jm} + \delta^{k}_{j}\nabla_{m}f + \delta^{k}_{m}\nabla_{j}f - g_{jm}\nabla^{k}f \quad (4.3.1)$$

And

$$\tilde{\Gamma}^{m}_{ik} = \Gamma^{m}_{ik} + \delta^{m}_{i}\nabla_{k}f + \delta^{m}_{k}\nabla_{i}f - g_{ik}\nabla^{m}f \quad (4.3.2)$$

Expand the product

$$\tilde{\Gamma}^{k}_{jm}\tilde{\Gamma}^{m}_{ik} = (\Gamma^{k}_{jm} + \delta^{k}_{j}\nabla_{m}f + \delta^{k}_{m}\nabla_{j}f - g_{jm}\nabla^{k}f)\,(\Gamma^{m}_{ik} + \delta^{m}_{i}\nabla_{k}f + \delta^{m}_{k}\nabla_{i}f - g_{ik}\nabla^{m}f). \quad (4.3.3)$$

Terms include

$$\Gamma^{k}_{jm}.\,\Gamma^{m}_{ik} = \Gamma^{k}_{jm}\Gamma^{m}_{ik} \quad (4.i)$$

$$\Gamma^{k}_{jm}.\,(\delta^{m}_{i}\nabla_{k}f) = \Gamma^{k}_{ji}\nabla_{k}f \quad (4.ii)$$

$$\Gamma^{k}_{jm}.\,(\delta^{m}_{k}\nabla_{i}f) = \Gamma^{k}_{jk}\nabla_{i}f \quad (4.iii)$$

$$\Gamma^{k}_{jm}.\,(-g_{ik}\nabla^{m}f) = -\Gamma^{k}_{jm}g_{ik}\nabla^{m}f \quad (4.iv)$$

$$\delta^{k}_{j}\nabla_{m}f.\,(\Gamma^{m}_{ik}) = \Gamma^{m}_{ij}\nabla_{m}f \quad (4.v)$$

$$\delta^{k}_{j}\nabla_{m}f.\,(\delta^{m}_{i}\nabla_{k}f) = \nabla_{i}f\nabla_{j}f \quad (4.vi)$$

$$\delta^{k}_{j}\nabla_{m}f.\,(\delta^{m}_{k}\nabla_{i}f) = \nabla_{j}f\nabla_{i}f \quad (4.vii)$$

$$\delta^{k}_{j}\nabla_{m}f.\,(-g_{ik}\nabla^{m}f) = -g_{ij}\nabla_{m}f\nabla^{m}f \quad (4.viii)$$

$$\delta^{k}_{m}\nabla_{j}f.\,(\Gamma^{m}_{ik}) = \Gamma^{m}_{im}\nabla_{j}f \quad (4.ix)$$

$$\delta^{k}_{m}\nabla_{j}f.\,(\delta^{m}_{i}\nabla_{k}f) = \nabla_{j}f\nabla_{i}f \quad (4.x)$$

$$\delta^{k}_{m}\nabla_{j}f.\,(\delta^{m}_{k}\nabla_{i}f) = \nabla_{j}f\nabla_{i}f \quad (4.xi)$$

$$\delta^{k}_{m}\nabla_{j}f.\,(-g_{ik}\nabla^{m}f) = -\nabla_{j}f\nabla_{i}f \quad (4.xii)$$

$$-\,g_{jm}\nabla^{k}f.\,(\Gamma^{m}_{ik}) = -\Gamma^{m}_{ik}g_{jm}\nabla^{k}f \quad (4.xiii)$$

$$-\,g_{jm}\nabla^{k}f.\,(\delta^{m}_{i}\nabla_{k}f) = -g_{ji}\nabla_{k}f\nabla^{k}f \quad (4.xiv)$$

$$-\,g_{jm}\nabla^{k}f.\,(\delta^{m}_{k}\nabla_{i}f) = -g_{jk}\nabla^{k}f\nabla_{i}f = -\nabla_{j}f\nabla_{i}f \quad (4.xv)$$

$$-\,g_{jm}\nabla^{k}f.\,(-g_{ik}\nabla^{m}f.) = g_{jm}g_{ik}\nabla_{k}f\nabla_{m}f = \nabla_{i}f\nabla_{j}f \quad (4.xvi)$$

$$\tilde{\Gamma}^{k}_{jm}\tilde{\Gamma}^{m}_{ik} = \Gamma^{k}_{jm}\Gamma^{m}_{ik} + \Gamma^{k}_{ji}\nabla_{k}f + \Gamma^{k}_{jk}\nabla_{i}f + \Gamma^{m}_{ij}\nabla_{m}f + \Gamma^{m}_{im}\nabla_{j}f + 3\nabla_{i}f\nabla_{j}f - g_{ik}\Gamma^{k}_{jm}\nabla^{m}f - g_{jm}\Gamma^{m}_{ik}\nabla^{k}f - g_{ij}\nabla_{m}f\nabla^{m}f - g_{ji}\nabla_{k}f\nabla^{k}f - 2\nabla_{j}f\nabla_{i}f + 2\,\nabla_{i}f\nabla_{j}f. \quad (4.3.4)$$

Simplify

$$\tilde{\Gamma}^{k}_{jm}\tilde{\Gamma}^{m}_{ik} = \Gamma^{k}_{jm}\Gamma^{m}_{ik} + \Gamma^{k}_{ji}\nabla_{k}f + \Gamma^{k}_{jk}\nabla_{i}f + \Gamma^{m}_{ij}\nabla_{m}f + \Gamma^{m}_{im}\nabla_{j}f + 3\nabla_{i}f\nabla_{j}f - g_{ik}\Gamma^{k}_{jm}\nabla^{m}f - g_{jm}\Gamma^{m}_{ik}\nabla^{k}f - 2\,g_{ji}\nabla_{k}f\nabla^{k}f. \quad (4.3.5)$$

4.4 Step 4: Simplify $\tilde{\Gamma}^{k}_{km}\tilde{\Gamma}^{m}_{ij}$ – $\tilde{\Gamma}^{k}_{jm}\tilde{\Gamma}^{m}_{ik}$

$$\tilde{\Gamma}^{k}_{km}\tilde{\Gamma}^{m}_{ij} - \tilde{\Gamma}^{k}_{jm}\tilde{\Gamma}^{m}_{ik} = [\Gamma^{k}_{km}\Gamma^{m}_{ij} + \Gamma^{k}_{ki}\nabla_{j}f + \Gamma^{k}_{kj}\nabla_{i}f - g_{ij}\Gamma^{k}_{km}\nabla^{m}f + n\Gamma^{m}_{ij}\nabla_{m}f + 2n\nabla_{i}f\nabla_{j}f - ng_{ij}\nabla_{m}f\nabla^{m}f] - [\Gamma^{k}_{jm}\Gamma^{m}_{ik} + \Gamma^{k}_{ji}\nabla_{k}f + \Gamma^{k}_{jk}\nabla_{i}f + \Gamma^{m}_{ij}\nabla_{m}f + \Gamma^{m}_{im}\nabla_{j}f + 3\nabla_{i}f\nabla_{j}f - g_{ik}\Gamma^{k}_{jm}\nabla^{m}f - g_{jm}\Gamma^{m}_{ik}\nabla^{k}f - 2\,g_{ji}\nabla_{k}f\nabla^{k}f] \quad (4.4.1)$$

$$\tilde{\Gamma}^k_{km}\tilde{\Gamma}^m_{ij} - \tilde{\Gamma}^k_{jm}\tilde{\Gamma}^m_{ik} = [\Gamma^k_{km}\Gamma^m_{ij} - \Gamma^k_{jm}\Gamma^m_{ik}] + [\Gamma^k_{ki}\nabla_j f + \Gamma^k_{kj}\nabla_i \mathrm{f} - \Gamma^k_{ji}\nabla_k f - \Gamma^k_{jk}\nabla_i f - \Gamma^m_{ij}\nabla_m f - \Gamma^m_{im}\nabla_j f + n\Gamma^m_{ij}\nabla_m f] + [g_{ik}\Gamma^k_{jm}\nabla^m f + g_{jm}\Gamma^m_{ik}\nabla^k f - g_{ij}\Gamma^k_{km}\nabla^m f] + [2\, g_{ji}\nabla_k f\nabla^k f - n g_{ij}\nabla_m f\nabla^m f] + [2n\nabla_i f\nabla_j f - 3\nabla_i f\nabla_j f] \tag{4.4.2}$$

$$\tilde{\Gamma}^k_{km}\tilde{\Gamma}^m_{ij} - \tilde{\Gamma}^k_{jm}\tilde{\Gamma}^m_{ik} = \Gamma^k_{km}\Gamma^m_{ij} - \Gamma^k_{jm}\Gamma^m_{ik} + (n-2)\Gamma^m_{ij}\nabla_m f + \left[g_{ik}\Gamma^k_{jm}\nabla^m f + g_{jm}\Gamma^m_{ik}\nabla^k f - g_{ij}\Gamma^k_{km}\nabla^m f\right] - \left[(n-2)g_{ij}\nabla_k f\nabla^k f\right] + \left[(2n-3)\nabla_i f\nabla_j f\right] \tag{4.4.3}$$

$$\tilde{\Gamma}^k_{km}\tilde{\Gamma}^m_{ij} - \tilde{\Gamma}^k_{jm}\tilde{\Gamma}^m_{ik} = \Gamma^k_{km}\Gamma^m_{ij} - \Gamma^k_{jm}\Gamma^m_{ik} + [(n-2)\Gamma^m_{ij}\nabla_m f] + [g_{ik}\Gamma^k_{jm}\nabla^m f + g_{jm}\Gamma^m_{ik}\nabla^k f - g_{ij}\Gamma^k_{km}\nabla^m f] - [(n-2)\, g_{ij}\nabla_k f\nabla^k f] + [(2n-3)\nabla_i f\nabla_j f] \tag{4.4.4}$$

4.5 Step 5: Combine all terms to generate $\tilde{\mathrm{R}}_{ij}$ from prior derivations, the nonlinear terms are:

$$\tilde{\mathrm{R}}_{ij} = \partial_k\tilde{\Gamma}^k_{ij} - \partial_j\tilde{\Gamma}^k_{ik} + \tilde{\Gamma}^k_{km}\tilde{\Gamma}^m_{ij} - \tilde{\Gamma}^k_{jm}\tilde{\Gamma}^m_{ik}. \tag{4.5.1}$$

And

$$R_{ij} = \partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} + \Gamma^k_{km}\Gamma^m_{ij} - \Gamma^k_{jm}\Gamma^m_{ik}. \tag{4.5.2}$$

$$\partial_k\tilde{\Gamma}^k_{ij} - \partial_j\tilde{\Gamma}^k_{ik} = \partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} + \nabla_i\nabla_j f - (n-1)\nabla_j\nabla_i f - g_{ij}\nabla_k\nabla^k f, \tag{4.5.3}$$

Combining all the collected contributions, the transformed Ricci tensor becomes:

$$\tilde{\mathrm{R}}_{ij} = R_{ij} + \nabla_i\nabla_j f - (n-1)\nabla_j\nabla_i f - g_{ij}\nabla_k\nabla^k f + [(2n-3)\nabla_i f\nabla_j f - (n-2)\, g_{ij}\nabla_k f\nabla^k f + C, \tag{4.5.4}$$

Where C are terms involving the Christoffel symbols

$$C = (n-2)\,\Gamma^m_{ij}\nabla_m f + g_{ik}\Gamma^k_{jm}\nabla^m f + g_{jm}\Gamma^m_{ik}\nabla^k f - g_{ij}\Gamma^k_{km}\nabla^m f. \tag{4.5.5}$$

In general, the transformed Ricci tensor is explicitly given as:

$$\tilde{\mathrm{R}}_{ij} = R_{ij} + \nabla_i\nabla_j f - (n-1)\nabla_j\nabla_i f - g_{ij}\nabla_k\nabla^k f + [(2n-3)\nabla_i f\nabla_j f - (n-2)\, g_{ij}\nabla_k f\nabla^k f + (n-2)\,\Gamma^m_{ij}\nabla_m f + g_{ik}\Gamma^k_{jm}\nabla^m f + g_{jm}\Gamma^m_{ik}\nabla^k f - g_{ij}\Gamma^k_{km}\nabla^m f. \tag{4.5.6}$$

Hence the proof.

4.6 Conclusion

This theorem provides a rigorous framework to understand how curvature properties of a manifolds transform under conformal scaling. The detailed expression is essential in areas like differential geometry, GR, and CFT. Each term has specific geometric meaning, helping to analyze distortions introduced by the scaling function f.

5. Conformal Transformation of Scalar Curvature Tensor, *R*.

Definition 5.1 (Scalar Curvature Tensor). The scalar curvature Tensor R of a Riemannian manifold (M, g) of dimension n is the trace of the Ricci tensor with respect to the metric defined as:

$$R = g^{ij}R_{ij}. \tag{5.1.1}$$

Definition 5.2 (Transformed scalar curvature tensor.) Under a conformal transformation

$\tilde{g}_{ij} = e^{2f}g_{ij}$,

The transformed scalar curvature tensor $\tilde{R}$, of the original scalar curvature tensor R, is given by:

$$\tilde{R} = \tilde{g}^{ij}\tilde{R}_{ij}. \tag{5.2.1}$$

Where $\tilde{R}_{ij}$ is the transformed Ricci tensor

Theorem 5.2 Let (M, g) be a Riemannian manifold of dimension n with scalar curvature tensor R. Under a conformal transformation $\tilde{g}_{ij} = e^{2f}g_{ij}$, the transformed scalar curvature $\tilde{\mathrm{R}}$ is given as follows:

$$\tilde{R} = e^{-2f}[R - 2(n-1)\nabla_k\nabla^k f - (n-1)(n-3)\nabla_k\nabla^k f + (n-2)g^{ij}\Gamma^m_{ij}\nabla_m f - (n-2)\,\Gamma^k_{km}\nabla^m f] \tag{5.2.2}$$

Proof.

The scalar curvature is the trace of the Ricci tensor. For the transformed metric $\tilde{g}_{ij} = e^{2f} g_{ij}$, the inverse metric is $\tilde{g}^{ij} = e^{-2f} g^{ij}$, and the transformed scalar curvature is:

$$\tilde{R} = \tilde{g}^{ij} \tilde{R}_{ij} = e^{-2f} g^{ij} \tilde{R}_{ij}. \quad (5.2.3)$$

Using equation (4.5.6), equation (5.2.3) reduces to

$$\tilde{R} = e^{-2f} g^{ij} [R_{ij} + \nabla_i \nabla_j f - (n-1) \nabla_j \nabla_i f - g_{ij} \nabla_k \nabla^k f + [(2n-3) \nabla_i f \nabla_j f -$$

$$(n-2) g_{ij} \nabla_k f \nabla^k f + (n-2) \Gamma_{ij}^m \nabla_m f + g_{ik} \Gamma_{jm}^k \nabla^m f + g_{jm} \Gamma_{ik}^m \nabla^k f - g_{ij} \Gamma_{km}^k \nabla^m f]. \quad (5.2.4)$$

$$\tilde{R} = e^{-2f} [g^{ij} R_{ij} + g^{ij} [\nabla_i \nabla_j f - (n-1) \nabla_j \nabla_i f - g_{ij} \nabla_k \nabla^k f] + g^{ij} [(2n-3) \nabla_i \nabla_j f - (n-2) g_{ij} \nabla_k \nabla^k f] + g^{ij} (n-2) \Gamma_{ij}^m \nabla_m f + g^{ij} [g_{ik} \Gamma_{jm}^k \nabla^m f + g_{jm} \Gamma_{ik}^m \nabla^k f - g_{ij} \Gamma_{km}^k \nabla^m f]] \quad (5.2.5)$$

$$\tilde{R} = e^{-2f} \left[R + [\nabla_k \nabla^k f - (n-1) \nabla_k \nabla^k f - n \nabla_k \nabla^k f] + [(2n-3) \nabla_k \nabla^k f - (n-2) n \nabla_k \nabla^k f] + g^{ij} (n-2) \Gamma_{ij}^m \nabla_m f + \left[\delta_k^j \Gamma_{jm}^k \nabla^m f + \delta_m^i \Gamma_{ik}^m \nabla^k f - n \Gamma_{km}^k \nabla^m f\right]\right] \quad (5.2.6)$$

$$\tilde{R} = e^{-2f} \left[R + [-2(n-1) \nabla_k \nabla^k f - (n-1)(n-3) \nabla_k \nabla^k f] + (n-2) g^{ij} \Gamma_{ij}^m \nabla_m f + \left[\Gamma_{km}^k \nabla^m f + \Gamma_{km}^k \nabla^m f - n \Gamma_{km}^k \nabla^m f\right]\right] \quad (5.2.7)$$

$$\tilde{R} = e^{-2f} [R + [-2(n-1) \nabla_k \nabla^k f - (n-1)(n-3) \nabla_k \nabla^k f] + (n-2) g^{ij} \Gamma_{ij}^m \nabla_m f + [\Gamma_{km}^k \nabla^m f - (n-1) \Gamma_{km}^k \nabla^m f]] \quad (5.2.8)$$

$$\tilde{R} = e^{-2f} [R - 2(n-1) \nabla_k \nabla^k f - (n-1)(n-3) \nabla_k \nabla^k f + (n-2) g^{ij} \Gamma_{ij}^m \nabla_m f -$$

$$(n-2) \Gamma_{km}^k \nabla^m f] \quad (5.2.9)$$

This completes the proof

5.3 Conclusion

The scalar curvature under a conformal transformation involves contributions from:

- The original curvature R
- The Laplacian of the conformal factor f which encodes how rapidly the metric is scaled.
- The gradient of f, which measures how non-uniform the scaling is across the manifold.

5.4 Geometric interpretation of scalar curvature under conformal transformation in equation (5.2.9) shows how dimensionality $(n = 1, n = 2, n = 3)$ uniquely affects the behavior of this scalar curvature. This is summarized in the following two theorems.

5.5 Conformal Scalar Curvature Transformations by Dimension

Theorem 5.5 (Scalar Curvature in Dimension N=1). Let (M, g) be a Riemannian manifold of dimension n, and let g be the metric tensor being transformed conformally as $\tilde{g}_{ij} = e^{2f} g_{ij}$.

Then conformal transformation has no geometric effect on the scalar curvature for manifold of dimension $n = 1$.

Proof.

In dimension N=1, a Riemannian manifold is a curve, and the metric is locally $g_{11} = 1$.

$$\tilde{R} = e^{-2f} [R - 2(1-1) \nabla_k \nabla^k f - (1-1)(n-3) \nabla_k \nabla^k f + (1-2) g^{ij} \Gamma_{ij}^m \nabla_m f - (1-2) \Gamma_{km}^k \nabla^m f]$$

$$\tilde{R} = e^{-2f} [0 - 0 - 0 + 0 - 0] = 0 \quad (5.5.1)$$

5.5.1 Geometric interpretation for N=1

- In one dimension, there is no intrinsic curvature, as the scalar curvature R identically vanishes.
- Conformal transformations do not change this, as the curvature of a one-dimension manifold is trivial.

Theorem 5.6 (Scalar Curvature in Dimension N=2). Let (M, g) be a Riemannian manifold of dimension $n = 2$, and let the metric transform conformally as $\tilde{g}_{ij} = e^{-2f} g_{ij}$. The transformed scalar curvature depends entirely on the Laplacian of f and the intrinsic curvature of the manifold:

$$\tilde{R} = e^{-2f} [R - 2\nabla_k \nabla^k f]. \quad (5.6.1)$$

Proof

For $n = 2$, the scalar curvature transformation under $\tilde{g}_{ij} = e^{-2f} g_{ij}$ is derived from the general formula (equation) (5.2.9)

$$\tilde{R} = e^{-2f}[R - 2(n-1)\nabla_k\nabla^k f - (n-1)(n-3)\nabla_k\nabla^k f + (n-2)g^{ij}\Gamma_{ij}^{m}\nabla_m f - (n-2)\Gamma_{km}^{k}\nabla^m f]$$

Substitute $n = 2$:

$$\tilde{R} = e^{-2f}[R - 2(2-1)\nabla_k\nabla^k f - (2-1)(2-3)\nabla_k\nabla^k f + (2-2)g^{ij}\Gamma_{ij}^{m}\nabla_m f -$$

$$(2-2)\Gamma_{km}^{k}\nabla^m f] \quad (5.6.2)$$

$$\therefore \tilde{R} = e^{-2f}[R - 2\nabla_k\nabla^k f - (-1)(-1)\nabla_k\nabla^k f + 0 - 0] \quad (5.6.3)$$

The term involving $n = 2$ vanish because $n - 2 = 0$. The term $-(n-1)(n-3)\nabla_k\nabla^k f$ also vanishes because $(n-3) = -1$, leaving no contribution. The results are:

$$\tilde{R} = e^{-2f}[R - 2\nabla_k\nabla^k f]. \square \quad (5.6.4)$$

5.6.5 Geometric interpretation for $n = 2$

- In two dimensions, the scalar curvature R is proportional to the Gaussian curvature, which measures the intrinsic curvature of a two-dimension surface.
- The term $-2\nabla_k\nabla^k f$ shows how the Laplacian of the conformal factor f modifies the scalar curvature. This reflects the stretching or compression caused by the conformal transformation.
- This case is foundational in complex geometry and two-dimension conformal field theory, where conformal transformations are angle-preserving but alter lengths and areas.
- This case is foundational in complex geometry and two-dimension conformal field theory, where conformal transformations are angle-preserving but alter lengths and areas.

Theorem 5.7 (Scalar Curvature in Dimension $n \geq 3$). Let (M, g) be a Riemannian manifold of dimension $n \geq 3$, and suppose the metric transforms conformally as $\tilde{g}_{ij} = e^{2f} g_{ij}$, where f is a (conformal factor) smooth function on M. While the conformal function e^{-2f} is independent of the dimension n, the effect on the scalar curvature depends critically on n. Specifically, the transformation involves the conformal factor f, its gradients, higher-order curvature terms, and the Christoffel symbols, with the precise impact varying with n:

$$\tilde{R} = e^{-2f}[R - 4\nabla_k\nabla^k f + g^{ij}\Gamma_{ij}^{m}\nabla_m f - \Gamma_{km}^{k}\nabla^m f]. \quad (5.7.1)$$

Proof.

For $\tilde{g}^{ij} = e^{-2f} g^{ij}$, the transformed scalar curvature is:

$$\tilde{R} = \tilde{g}^{ij}\tilde{R}_{ij} = e^{-2f} g^{ij}\tilde{R}_{ij}.$$

$$\tilde{R} = e^{-2f}[R - 2(3-1)\nabla_k\nabla^k f - (3-1)(3-3)\nabla_k\nabla^k f + (3-2)g^{ij}\Gamma_{ij}^{m}\nabla_m f -$$

$$(3-2)\Gamma_{km}^{k}\nabla^m f] \quad (5.7.2)$$

$$\tilde{R} = e^{-2f}[R - 4\nabla_k\nabla^k f - 0 + g^{ij}\Gamma_{ij}^{m}\nabla_m f - \Gamma_{km}^{k}\nabla^m f]$$

$$\therefore \tilde{R} = e^{-2f}[R - 4\nabla_k\nabla^k f + g^{ij}\Gamma_{ij}^{m}\nabla_m f - \Gamma_{km}^{k}\nabla^m f] \quad (5.7.3)$$

This completes the proof.

5.7.4 Geometric interpretation for N=3

- In three dimensions, the scalar curvature measures how the manifold bends in three-dimension space. Its encodes both the intrinsic curvature and the influence of f.
- The Laplacian term $-4\nabla_k\nabla^k f$ dominates, showing the effects of f on volume changes.
- The Christoffel-dependent terms, $g^{ij}\Gamma_{ij}^{m}\nabla_m f$ and $-\Gamma_{km}^{k}\nabla^m f$, reflect directional variations introduced by the conformal transformation.
- For dimensions $n = 3$, the scalar curvature is influenced by the conformal factor, which introduces dependence on-order curvature terms, including the gradients of f and the Christoffel symbols.

6. Conclusion
These theorems underscores the critical role of dimensionality in the interplay between geometry and conformal transformations.

7. References

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