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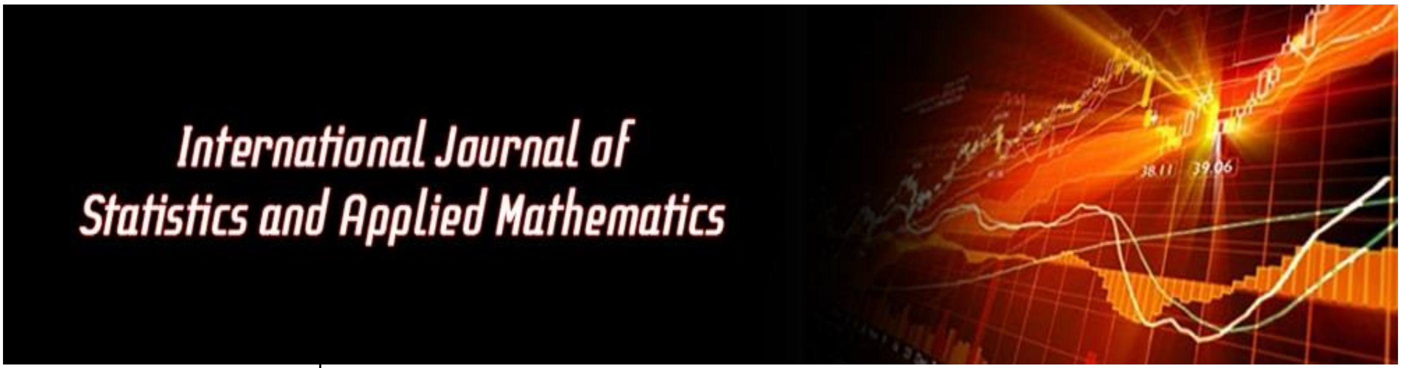

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On lorentzian para-kenmotsu manifolds satisfying W_i curvature tensor

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Abstract
This paper discusses the properties of W_7-curvature tensors on L P Kenmotsu manifolds. Properties of this Curvature tensor under various conditions on these manifolds are explored and their geometric implications are examined. The study includes investigations of W_7-flatness, ξ-W_7-flatness, ϕ-W_7 flatness, W_7-Semisymmetric, $W_7 \cdot R=0$.

Keywords: Lorentzian Para-Kenmotsu, W_i curvature tensor, paper discusses, riemannian manifold

1. Introduction

Abdul Haseeb and Rajendra Prasad in 2018 introduced Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifolds, studying ϕ-semisymmetric LP-Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons. 1970, [5] Pokhariyal and Mishra introduced new tensor fields on a Riemannian manifold, called the Weyl-projective curvature tensor of type (1, 3) and the tensor field E. The concept of W_7-curvature tensor was defined by Tripathi and Gupta [15] of an n (where $n=2m+1$)-dimensional Riemannian manifold are, respectively, defined as

$$W_7(X,Y,Z,T)=R(X,Y,Z,T)+\frac{1}{n-1}[g(Y,Z)S(X,T)-g(X,T)S(Y,Z)] \qquad (1)$$

$$W_7(X,Y)Z=R(X,Y)Z+\frac{1}{n-1}[g(Y,Z)QX-S(Y,Z)X]$$

For all $(X, Y, Z) \in_X (M)$. Where R represents the curvature tensor and S corresponds to the Ricci tensor of the manifold.

2. Preliminaries

A differentiable manifold M admitting a (ϕ, ξ, η, g), (1,1) tensor field ϕ, contravariant vector field ξ, a 1-form η and the Lorentzian metric g is called Lorentzian almost Paracontact manifold [7] if it satisfies:

$$\phi^2 X=X+\eta(X)\xi \qquad (2)$$

$$\eta(\xi)=-1 \qquad (3)$$

$$\phi\xi=0 \qquad (4)$$

$$rank\phi=n-1 \qquad (5)$$

$$\phi(X,Y)=\phi(Y,X) \qquad (6)$$

Where,

$$\phi(X,Y)=g(X,\phi Y) \qquad (7)$$

In the Lorentzian Para Kenmotsu manifold we have

$g(X, \phi Y) = g(\phi X, Y)$ (8)

$g(X, \xi) = \eta(X)$ (9)

A Lorentzian almost Paracontact manifold M is called Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifold (2018) [3] if

$(\nabla_X \phi) Y = -g(\phi X, Y)\xi - \eta(Y)\phi X,$ (10)

On the LP-Kenmotsu manifold, the following relations hold [4]

$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi$ (11)

$(\nabla_X \eta) Y = -g(X,Y) - \eta(X)\eta(Y)$ (12)

$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X$ (13)

$R(\xi, X)\xi = X + \eta(X)\xi = -\nabla_X \xi,$ (14)

$R(X,Y)\xi = \eta(Y)X - \eta(Y)X,$ (15)

$S(X,\xi) = (n-1)\eta(X),$ (16)

$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y),$ (17)

$Q\xi = (n-1)\xi$ (18)

$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y),$ (19)

∀ vector fields *X,Y,Z* on *M* and where *S* is Ricci Tensor and *Q*, Ricci operator and R Curvature tensor with respect to Levi-Civita connection ∇.

A Lorentzian Para-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci-tensor $S(X,Y)$ is of the form (2021) [7]
$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$ (20)

Where a and b are scalar functions on M. In particular, if $b = 0$, then the manifold is said to be an Einstein manifold.

3. A W_7-flat L.P Kenmotsu manifold

Definition 3.1 An n-dimensional L. P. Kenmotsu manifold is said to be W_7-flat if its W_7-curvature tensor satisfies the following condition

$W_7(X,Y)Z = 0$

Theorem 3.1; W_7-flat L.P-Kenmotsu manifold is a flat manifold.

Proof.
Suppose the LP-Kenmotsu manifold is W_7-flat, then the following hold

$W_7(X,Y)Z = 0$

$W_7(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(Y,Z)QX - S(Y,Z)X] = 0$ (3a)

$(n-1)R(X,Y)Z = S(Y, Z)X - (n-1)(g(Y, Z)X$

$(n-1)R(X,Y)Z = (n-1)(g(Y, Z)X - (n-1)(g(Y, Z)X$ (3b)

Equation (3b) reduces to

$(n-1)R(X,Y)Z = 0$ (3c)

Equation (3c) yields

$R(X,Y)Z=0$ (3d)

Hence proved.

4. A ξ- W_7-flat Lorentzian Para Kenmotsu manifold

Definition 4.1: An n-dimensional LP-Kenmotsu manifold is said to be ξ- W_7-flat if its

W_7-curvature tensor satisfies the following condition.
$W_7(X,Y)\xi=0$

Theorem 4.1: An ξ- W_7 flat Lorentzian Para Kenmotsu manifold is a special type of η-Einstein manifold.
Let us consider

$$W_7(X, Y)\xi=R(X, Y)\xi+\frac{1}{n-1}\{g(Y, \xi)QX-S(Y, \xi)X] \quad (4a)$$

$$0=\eta(Y)X-\eta(X)Y+g(Y,\xi)X-g(Y,\xi)X \quad (4b)$$

But
$S(Y,\xi)X=(n-1)g(Y,\xi)X$

$R(X,Y)\xi=\eta(Y)X-\eta(X)Y$

$$\eta(X)Y=\eta(Y)X \quad (4c)$$

$$\eta(X)g(Y,U)=\eta(Y)g(X,U) \quad (4d)$$

Setting $Y=\xi$

$$\eta(X)g(\xi,U)=\eta(\xi)g(X,U) \quad (4e)$$

$\eta(X)\eta(U)=-g(X,U)$

$g(X,U)=-\eta(X)\eta(U)$

$$S(X,U)=-(n-1)\eta(X)\eta(U) \quad (4f)$$

Hence (4f) is a special η-Einstein manifold

5. W_7-Semisymmetric Lorentzian Para Kenmotsu manifold

Definition 5.1 An n-dimensional W_7-Lorentzian-Para-Kenmotsu manifold M is is called Semisymmetric if for all vector fields X,Y, Z, U, V on M, the following holds:

$(R(X,Y)\cdot R)(U,V)Z=0.$ (5a)

Where $R(X, Y)$ serves as a derivation on the curvature tensor W_7.

Theorem 5.1 Any Lorentzian Para-Kenmotsu manifold is **s**emisymmetric.

Proof of Theorems

Proof of Theorem 5.1

Proving that

$$(R(X,Y)\cdot R)(U,V)Z=0. \quad (5b)$$

Hence, expanding $(R(X,Y)\cdot R)(U,V)Z$ we get

$$(R(X,Y)\cdot R)(U,V)Z=R(X,Y)R(U,V)Z-R(R(X,Y)U,V)Z-R(U,R(X,Y)V)Z-R(U,V)R(X,Y)Z \quad (5c)$$

Putting $U=\xi$ in 5.c above gives

$$(R(X,Y)R)\cdot R)(\xi,V)Z=R(X,Y)R(\xi,V)Z-R(R(X,Y)\xi,V)Z-R(\xi,R(X,Y)V)Z-R(\xi,V)R(X,Y)Z \quad (5d)$$

Simplifying each of the four terms in (5d) separately yields:

First term: $R(X,Y)R(\xi,V)Z$

$R(X,Y)R(\xi,V)Z=R(X,Y)[g(V,Z)\xi-\eta(Z)V]$

$R(X,Y)R(\xi,V)Z=[g(Y,g(V,Z)\xi)X-g(X,g(V,Z)\xi)Y-\eta(Z)g(Y,V)X+\eta(Z)g(X,V)Y$

$$R(X,Y)R(\xi,V)Z=\eta(Y)g(V,Z)X-\eta(X)g(V,Z)Y-\eta(Z)g(Y,V)X+\eta(Z)g(X,V)Y \quad (5e)$$

Second Term: $R(R(X,Y)\xi,V)Z$

$R(R(X,Y)\xi,V)Z=R([\eta(Y)X-\eta(X)Y],V)Z$

$=g(V,Z)\eta(Y)X-g(V,Z)\eta(X)Y-g([\eta(Y)X-\eta(X)Y],Z)V$

$$\therefore R(R(X,Y)\xi,V)Z=\eta(Y)g(V,Z)X-\eta(X)g(V,Z)Y-\eta(Y)g(X,Z)V+\eta(X)g(Y,Z)V \quad (5f)$$

Third term: $R(\xi,R(X,Y)V)Z$

$R(\xi,R(X,Y)V)Z=R(\xi,[g(Y,V)X-g(X,V)Y])Z$

$=g([g(Y,V)X-g(X,V)Y],Z)\xi-\eta(Z)g(Y,V)X+\eta(Z)g(X,V)Y$

$\therefore R(\xi,R(X,Y)V)Z=g(X,Z)g(Y,V)\xi-g(Y,Z)g(X,V)\xi-\eta(Z)g(Y,V)X+\eta(Z)g(X,V)Y$ (5g)

Fourth term: $R(\xi,V)R(X,Y)Z$

$R(\xi,V)R(X,Y)Z=g(V,R(X,Y)Z)\xi-\eta(R(X,Y)Z)V$

$=g(V,[g(Y,Z)X-g(X,Z)Y])\xi-\eta([g(Y,Z)X-g(X,Z)Y])V$

$\therefore R(\xi,V)R(X,Y)Z=g(V,X)g(Y,Z)\xi-g(V,Y)g(X,Z)\xi-\eta(X)g(Y,Z)V+\eta(Y)g(X,Z)V$ (5h)

Putting equations (5e), 5f), (5g) and (5h) in equation 5d) gives
$\therefore (R(X,Y)\cdot R)(\xi,V)Z=0$ (5.i)

This completes the proof that Lorentzian Para-Kenmotsu is a semisymmetric manifold.

6. W_7 – Para Kenmotsu Manifold satisfying the condition $W_7\cdot R=0$

Definition 6.1 A-Lorentzian-Para-Kenmotsu manifold M is said to satisfy $W_7.R=0$ condition if W_7

$$(W_7(U,V)\cdot R)(X,Y)Z=0 \quad (6a)$$

For any vector fields X, Y, Z, U, V on M.

Theorem 6.1, A W_7-Lorentzian-para-Kenmotsu manifold satisfies the condition $W_7\cdot R=0$

Proof of theorem 6.1
Proving that the relation (6a) holds for a W_7-Lorentzian para-Kenmotsu Manifold
The above equation (6.a) can be written as follows:

$(W_7(U,V)\cdot R)(X,Y)Z=W_7(U,V)R(X,Y)Z-R(W_7(U,V)X,Y)Z-R(X,W_7(U,V)Y)Z$

$$-R(X,Y)W_7(U,V)Z \quad (6.b)$$

Putting $U=\xi$ in (6.2) gives
$$(W_7(\xi,V)\cdot R)(X,Y)Z=W_7(\xi,V)R(X,Y)Z-R(W_7(\xi,V)X,Y)Z-R(X,W_7(\xi,V)Y)Z-R(X,Y)W_7(\xi,V)Z \quad (6c)$$

But
$R(X,Y)Z=g(Y,Z)X-g(X,Z)Y$

$W_7(\xi,V)W=g(V,W)\xi-g(\xi,W)V$

$W_7(\xi,V)W=g(V,W)\xi-\eta(W)V$ (6d)

Computing the four terms in (6c) separately gives
First term: $W_7(\xi,V)R(X,Y)Z$

From (6e), gives
$W_7(\xi,V)W=[g(V,[g(Y,Z)X-g(X,Z)Y])\xi-\eta([g(Y,Z)X-g(X,Z)Y])V]$

$W_7(\xi,V)R(X,Y)Z=g(V,X)g(Y,Z)\xi-g(V,Y)g(X,Z)\xi-\eta(X)g(Y,Z)V-\eta(Y)g(X,Z)V$ (6f)

Second Term: $R(W_7(\xi,V)X,Y)Z$

$R([g(V,X)\xi-g(\xi,X)V],Y)Z=g(Y,Z)[g(V,X)\xi-g(\xi,X)V]-g([g(V,X)\xi-g(\xi,X)V],Z)Y$

$R([g(V,X)\xi-g(\xi,X)V],Y)Z=g(Y,Z)g(V,X)\xi-g(Y,Z)g(\xi,X)V-g(\xi,Z)g(V,X)Y+g(V,Z)g(\xi,X)Y$

$W_7(\xi,V)R(X,Y)Z=g(Y,Z)g(V,X)\xi-\eta(X)g(Y,Z)V-\eta(Z)g(V,X)Y+\eta(X)g(V,Z)Y$ (6g)

Third Term: $R(X, W_7(\xi,V)Y)Z$

$R(X, W_7(\xi,V)Y)Z=g(W_5(\xi,V)Y,Z)X-g(X,Z)W_5(\xi,V)Y$

$R(X, W_7(\xi,V)Y)Z=g([g(V,Y)\xi-\eta(Y)V],Z)X-g(X,Z)[g(V,Y)\xi-\eta(Y)V]$

$R(X, W_7(\xi,V)Y)Z=\eta(Z)g(V,Y)X-\eta(Y)g(V,Z)X-g(X,Z)g(V,Y)\xi+\eta(Y)g(X,Z)V]$ (6.h)

Fourth Term: $R(X,Y)W_7(\xi,V)Z$

$W_7(\xi,V)Z=g(V,Z)\xi-\eta(Z)V$

$R(X,Y)W_7(\xi,V)Z=g(Y,[g(V,Z)\xi-\eta(Z)V])X-g(X,[g(V,Z)\xi-\eta(Z)V])Y$

$R(X,Y)W_7(\xi,V)Z=\eta(Y)g(V,Z)X-\eta(Z)g(Y,V)X-\eta(X)g(V,Z)Y+\eta(Z)g(X,V)Y$ (6i)

Plugging in equations (6e), (6g), (6h) and (6i) together gives

$W_7\cdot R=0$

This completes the proof of the theorem (6.1)

7. A ϕ-W_7-LP-Kenmotsu manifold
Definition 7.1: A LP-Kenmotsu manifold is said to be ϕ- W_7-flat if

W_7-curvature tensor satisfies the following condition
$W_7(X,Y).\phi)Z=W_7(X,Y)\phi Z-\phi(W_7(X,Y)Z=0$ (7a)

Theorem 7: A W_7-LP-Kenmotsu –manifold is a ϕ- W_7 *flat* manifold
Proof.

Consider ϕ-W_7 LP-Kenmotsu manifold, then the following hold
$W_7(X,Y)\phi Z-\phi(W_7(X,Y)Z$

$=R(X,Y)\phi Z-g(Y,Z)\phi X+g(X,Z)\phi Y+g(X,Y)\phi Z-g(Y,\phi Z)X-g(X,Y)\phi Z+g(Y,Z)\phi X$ (7b)

$=R(X,Y)\phi Z+g(X,Z)\phi Y-g(Y,\phi Z)X$

$R(X,Y)\phi Z=g(Y,\phi Z)X-g(X,Z)\phi Y$ (7.c)

$=g(Y,\phi Z)X-g(X,\phi Z)Y-g(Y,Z)\phi X+g(X,Z)\phi Y+g(X,Y)\phi Z-g(Y,\phi Z)X-g(X,Y)\phi Z+g(Y,Z)\phi X$ (7d)

$=-g(X,\phi Z)Y+g(X,Z)\phi Y$ (7e)

Let *Y*=*Z*=*ξ* in above equation yields

$W_7(X,\xi).\phi)\xi=0$

This completes the proof.

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