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Mittag-Leffler stability analysis of fractional differential equations with and without input using caputo derivative

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Abstract
In this paper, the stability analysis of the solutions of fractional differential equations has been investigated. Mittag-Leffler stability analysis of fractional differential equations with and without input is studied for various systems of fractional differential equations. The study employs Lyapunov characterization, Lyapunov direct method and a novel theorem to analyze the Mittag-Leffler stability of fractional differential equations with and without input. Examples have been given to illustrate the utilization of the provisions in analyzing the Mittag-Leffler stability of the solutions of fractional differential equations.

Keywords: Mittag-Leffler stability, Mittag-Leffler input stability, Lyapunov characterization, Lyapunov direct method, Caputo derivative

1. Introduction

Fractional calculus which is a generalization of the classical calculus has been studied for more than three centuries receiving attention due to its important role in modelling dynamics of various processes in most areas of science and engineering. Fractional derivative unarguably, has a long history beginning in 1695 when l'hospital asked Leibnitz and the community of mathematicians what $\frac{d^n y}{dx^n}$ would mean if n is not an integer. The answer to the question gave birth to the development of many fractional derivatives such as Riemann-Liouville derivative, Gronwald-Letnikov derivative, Caputo derivative, Comformable derivative, Atangana-Baleanu derivative, Atangana-Koca-Caputo derivative, Caputo-Fabrizio derivative, Caputo-Liouville etc. For more on types of derivatives, see [1-7].

The importance of studying and examining the stability of systems cannot be over-emphasized. Stability analysis is carried out on systems because of its time-serving and resources conservation benefits. Before analyzing, many physical systems are expressed or modelled as differential equations. The solutions of these differential equations are obtained and analysed with a view to establishing the stability status of the solutions. Many works focusing on stability analysis of differential equations for both classical and fractional order have been done for decades [8-18].

To determine the stability status of fractional differential equations with and without inputs, many stability notions are employed. Such notions include: Asymptotic stability, global asymptotic stability, local stability, Mittag-Leffler stability, fractional input stability, Mittag-Leffler input stability, conditional Mittag-Leffler input stability, practical stability, generalized Mittag-Leffler stability, global Mittag-Leffler stability and others. The Mittag-Leffler stability is one of the most important stability notions in fractional differential equations. Mittag-Leffler input stability is a special case of fractional input stability recently introduced in the literature of fractional differential equations. Mittag-Leffler stability with and without inputs has received attentions in recent years.

The following are some of the investigations: Omri F, [19] considered some classes of time-varying fractional systems and studied the problem of stabilization for these systems with norm-bounded controls. Time-varying Lyapunov functions were used in analyzing the Mittag-Leffler stability of these systems.

Elif *et al*. [20] considered the stability analysis of a neural network system and presented conditions that ensured the Mittag-Leffler stability of the equilibrium point obtained using Lyapunov direct method. Studies show that Lyapunov direct method is a sufficient condition which means that if the Lyapunov function candidate is found, the system is stable. Also, if the Lyapunov function candidate cannot be found, the system may be stable. Agarwal *et al*. [21] studied the stability of generalized proportional Caputo fractional differential equations by Lyapunov functions and provided sufficient conditions for stability and asymptotic stability. Basdouri *et al*. [22] examined the practical Mittag-Leffler stability of quasi-one-sided Lipschitz fractional order system. The study provided sufficient conditions for the practical Mittag-Leffler stability of the closed loop system with a linear and state feedback. Dong *et al*. [23] studied Mittag-Leffler stability of numerical solutions to linear homogeneous time fractional parabolic equations. The analysis proved that the strongly stable fractional linear multistep method combined with appropriate spatial discretization can accurately maintain the long-term optimal algebraic decay rate of the original continuous equation.

Wang *et al*. [24] analyzed the stability of fractional –order nonlinear systems with delay. The analysis proposed the Mittag-Leffler stability of time-delay system and introduce the fractional Lyapunov direct method by using properties of Mittag-Leffler function and Laplace transform. Sene [25] studied a particular class of fractional nonlinear systems with a Lyapunov characterization of the conditional Mittag-Leffler stability and conditional asymptotic stability of the fractional nonlinear systems with exogenous input. This work studies the properties of Mittag-Leffler functions and analyses the Mittag-Leffler stability of fractional differential equations with and without input as well as its applications.

2. Definitions and Preliminary Analysis

This section begins with some notations as will be used in the work.

Notation 2.1

The class PD function denotes the set of all continuous functions $\alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying. Satisfying $\alpha(0) = 0$, and $\alpha(s) > 0$ for all $s > 0$. A class K function is an increasing PD function. The class K_∞ represents the set of all unbounded K functions.

Notation 2.2

A continuous function $\beta: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class KL if $\beta(.,t) \in K$ for any $t \geq 0$, and $\beta(s,.)$ is non-increasing and tends to zero as its arguments tend to infinity.

Notation 2.3

Given $x \in R^n, \|x\|$ stands for its Euclidean norm. $\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}$

Notation 2.4

For a matrix $A, \lambda_{max}(A)$ *and* $\lambda_{min}(A)$ denote the maximal and minimal eigenvalue of A, respectively.

The following are some definitions that will be needed.

Definition 2.1

In the solutions of classical systems, one of the frequently used functions is the exponential function. Similarly, the Mittag-Leffler function is frequently used in the solutions of fractional-order systems. The Mittag-Leffler function is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

Where $\alpha > 0$, and $z \in \mathbb{C}$.

The Mittag-Leffler function with two parameters is used more often and is defined by the series.

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

Where the parameters $\alpha > 0, \beta \in \mathbb{R}$ and $z \in \mathbb{C}$. For $\beta = 1, E_\alpha(z) = E_{\alpha,1}(z)$.

If $\alpha = 1$ *and* $\beta = 1$, then $E_{1,1}(z) = e^x$.

Definition 2.2

The Caputo fractional derivative is given by

$$(D_\alpha^c f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds,$$

For all $t > 0$, where the order $\alpha \in (0,1)$, and $\Gamma(\ldots.)$ is the gamma function.

Definition 2.3

Given a function $f : [0, +\infty[\to \mathbb{R}$. Then, the Riemann-Liouville fractional derivative of f of order α is defined by

$$(D_\alpha^{RL} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

For $t > 0, and\ \alpha \in (0,1)$.

Definition 2.4

Let the function $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and locally Lipschitz with Lipschitz constant L, then.

$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$.

Definition 2.5

The class PD function denotes the set of all continuous functions $\alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\alpha(0) = 0$, and $\alpha(s) > 0$ for all $s > 0$. A class K function is an increasing PD function. The class K_∞ represents the set of all unbounded K functions.

Definition 2.6

A continuous function $\beta: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class KL if $\beta(.,t) \in K$ for any $t \geq 0$, and $\beta(s,.)$ is non-increasing and tends to zero as its arguments tend to infinity.

To study stability notion with input, it is important to examine some of the properties of input stability. When an input is introduced, a solution is generated. If the input introduced is convergent, then, the generated solution is also convergent. This property is known as Converging-Input-Converging-

State (CICS). If the input is bounded, the generated solution is bounded. This is known as Bounded-Input-Bounded-State (BIBS). Also, if the input is zero, then, the trivial solution of the fractional differential equation without input is stable. The above properties offer alternative method of analyzing stability notion with input. These properties will be used later for further analysis. The following stability notions are defined as follows.

Definition 2.7
The origin of the fractional differential equation (without input) defined by $D_\alpha^c x = f(x,0)$ is said to be Mittag-Leffler stable, if for any initial condition x_0, it solution satisfies.

$$\|x(t,x_0)\| \le [m(\|x_0\|)E_\alpha(\lambda(t-t_0)^\alpha)]^{\frac{1}{q}}$$

Where $q > 0, and\ m$ is locally Lipschitz on a domain contained in $\mathbb{R}^n$ with a Lipschitz constant K, and satisfies $m(0) = 0$.

Definition 2.8
The fractional differential equation defined by $D_\alpha^c x = f(x,u)$ is said to be Mittag-Leffler input stable if for any input $u \in \mathbb{R}^n$, there exists a class K_∞ function γ such that for any initial condition $x(t_0)$, its solution satisfies

$$\|x(t,x_0,u)\| \le [G(\|x_0\|)E_\alpha(\lambda(t-t_0)^\alpha)]^{\frac{1}{q}} + \gamma(\|u\|_\infty)$$

Where $G\ and\ q > 0$ are nonnegative constants.

Definition 2.9
The fractional differential equation defined by $D_\alpha^c x = f(x,u)$ is said to be fractional input stable if for any input $u \in \mathbb{R}^n$, there exists a class KL function μ, and a class K_∞ function γ such that for any initial condition $x(t_0)$, its solution satisfies

$$\|x(t,x_0,u)\| \le \mu(\|x_0\|, t-t_0) + \gamma(\|u\|_\infty)$$

Where γ is the asymptotic gain.

Definition 2.10
The trivial solution to system $D_\alpha^c x = f(t,x,0)$ is said to be stable if, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that for any initial condition $\|x_0\| < \delta$, the solution $x(t)$ of the system $D_\alpha^c x = f(t,x,0)$ satisfies the inequality $\|x_0\| < \epsilon$ for all $t > t_0$.
The trivial solution to system $D_\alpha^c x = f(t,x,0)$ is said to be asymptotically stable if it is stable and furthermore $\lim_{t\to\infty} x(t) = 0$.

3. Analysis of Mittag-Leffler stability with and without input

Consider the fractional differential equation represented by

$$D_\alpha^c x = Gx + Qu \tag{3.1}$$

Where $x \in \mathbb{R}^n$ is the state variable, $G \in \mathbb{R}^{n\times m}$, Q is n matrix in $\mathbb{R}^{n\times m}$ and $u \in \mathbb{R}^n$ is the input.

The solution of equation (3.1) is the following

$$x(t) = x_0 E_\alpha(G(t-t_0)^\alpha) + \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(G(t-s)^\alpha)Qu(s)ds \tag{3.2}$$

Applying the norm to the solution, we have

$$\|x(t)\| \le \|x_0\|\|E_\alpha(G(t-t_0)^\alpha)\| + \|Q\|\|u\|_\infty \int_{t_0}^t \left\|(t-s)^{\alpha-1}E_{\alpha,\alpha}(G(t-s)^\alpha)\right\|ds \tag{3.3}$$

Let the matrix G satisfies the Magtinon condition, then, there exists a positive constant H such that

$$\int_{t_0}^t \left\|(t-s)^{\alpha-1}E_{\alpha,\alpha}(G(t-s)^\alpha)\right\|ds \le H \tag{3.4}$$

Consequently, equation (3.3) becomes

$$\|x(t)\| \le \|x_0\|\|E_\alpha(G(t-t_0)^\alpha)\| + \|Q\|\|u\|_\infty H \tag{3.5}$$

Equation (3.5) is in the form of definition 2.8

Where $\|x_0\|\|E_\alpha(G(t-t_0)^\alpha)\| = \mu(\|x_0\|, t-t_0)\ and\ \gamma(\|u\|_\infty) = \|Q\|\|u\|_\infty H$.

Since Mittag-Leffler input stability is a special case of fractional input stability,if $\mu(\|x_0\|, t-t_0) = [K\|x_0\|\|E_\alpha(\lambda(t-t_0)^\alpha)\|]^{\frac{1}{q}}$, then, the solution of (3.1) is Mittag-Leffler input stable.

From equation (3.5), if $u = 0$, then the solution of the fractional differential equation is given by

$$\|x(t)\| \le \|x_0\|\|E_\alpha(G(t-t_0)^\alpha)\|$$

Therefore, the unforced equation (3.1), i.e. $D_\alpha^c x = Gx$ is Mittag-Leffler stable.
Another method of establishing Mittag-Leffler input stability of fractional differential equations with inputs is Lyapunov characterization. The following theorem which was first stated and proved by Sontag [26] for the case of integer differential equations is stated.

Theorem 3.1
If there exists a positive function $R^+ \times R^n \to R$ that is continuous and differentiable, a K_∞ function Ω_1, Ω_2, and class K function Ω, satisfying the following conditions:

- $\|x\|^q \le V(t,x) \le \Omega_1(\|x\|)$
- If for any $\|\boldsymbol{x}\| \ge \Omega_2(\|u\|)$

$$\Rightarrow D_\alpha^c V(t,x) \le -kV(t,x) + \Omega_3(\|u\|)$$

Where $q > 0$. Then, the fractional differential equation $D_\alpha^c x = f(x,u)$ is Mittag-Leffler input stable.

Proof
From conditions 1 and 2, it obvious that

$$\|x\|^q \le \Omega_3(\|u\|) \tag{3.6}$$

Where $\Omega_3(\|u\|) \in K_\infty$. Also, from (3), there exists a positive constant such that

$\|\boldsymbol{x}\| \ge \Omega_3(\|u\|)$,
$\Rightarrow D_\alpha^c V(t,x) \le -\Omega_3(\|x\|) \le -kV(t,x)$

It follows from (3.6) that
$\|x\|^q \le V(t,x)$

$$\|x\| \leq [V(t,x)]^{\frac{1}{q}} \quad (3.7)$$

Combining (3.6) and (3.7), we have the following

$$\|x\| \leq [V(t,x)]^{\frac{1}{q}} + \Omega_3(\|u\|).$$

Lyapunov Characterization

To analyze using Lyapunov characterization, we choose a Lyapunov candidate function

$$V(t,x) = x^T Px, where\ A^T P + PA = -Q\ and\ P = I_n$$

The time derivative of V along the solution path of (3.1) is given by

$$D_\alpha^c V(t,x) \leq 2x^T P D_\alpha^c x = [Ax + Bu]^T Px + x^T P[Ax + Bu]$$
$$= x^T(A^T P + PA)x + (Bu)^T Px + x^T P(Bu)$$

$$\leq -\lambda_{min}(Q)\|x\|^2 + 2\lambda_{max}(P)\|B\|\|u\|\|x\|$$

Choose $\theta \in (0,1)$ and let $k = \frac{2\|P\|\|B\|}{\lambda_{min}(Q)-\theta}$ and $\Omega_4(r) = kr$

If $\|x\| \geq \Omega_4(\|u\|)$, it implies that

$$D_\alpha^c V(t,x) \leq -\theta\|x\|^2 = -\theta V(t,x)$$

Thus, the given fractional differential equation is Mittag-Leffler input stable.

4. Bilinear Fractional Differential Equations

Consider the bilinear fractional differential equation defined by

$$D_\alpha^c x = Gx + Qux \quad (4.1)$$

Where $x \in \mathbb{R}^n$ is a state variable, where G is a matrix in where $\mathbb{R}^{n\times n}$, Q is a matrix in $\mathbb{R}^{n\times m}$, and $u \in \mathbb{R}$ is the input. We assume that the matrix G satisfies the classical Matignon condition $|\arg(\lambda(G))| > \frac{\alpha\pi}{2}$. If the input $u = 0$, equation (4.1) becomes

$$D_\alpha^c x = Gx \quad (4.2)$$

The solution of equation (4.2) is given by

$$x(t) = x_0 E_\alpha(G(t-t_0)^\alpha)$$

Therefore, the fractional differential equation (4.2) without input is Mittag-Leffler stable.

If $u \neq 0$, the solution of equation (4.1) is not generally Mittag-Leffler input stable even though it satisfies CICS and BIBS properties.

For example, consider

$$D_\alpha^c x = -3x + 4xu. \quad (4.3)$$

If $u(t) = -2$, the equation becomes $D_\alpha^c x = -11x$ and the solution is given by

$$x(t) = x_0 E_\alpha(-11(t-t_0)^\alpha)$$

The above solution is MLIS. On the other hand, if $u(t) = 2$, equation (4.3) becomes $D_\alpha^c x = 5x$. The solution is given by

$$x(t) = x_0 E_\alpha(5(t-t_0)^\alpha) \quad (4.4)$$

The solution of equation (4.4) is obviously divergent. Therefore, it is not MLIS.

5. Applications

Example 1

Consider the RC electrical circuit described by

$$D_\alpha^c x = -\frac{\sigma^{1-\alpha}}{RC}x + u \quad (5.1)$$

With the initial boundary condition defined by $x(0) = x_0$, where σ is associated with the temporal components in the differential equation and u represents the input. Let the Lyapunov candidate function be defined by $V(x) = \frac{1}{4}\|x\|^2$. The derivative of the Lyapunov function along the trajectories is given by

$$D_\alpha^c V(t,x) = -\frac{\sigma^{1-\alpha}}{RC}x^2 + xu$$

$$\leq -\frac{\sigma^{1-\alpha}}{RC}\|x\|^2 + \frac{1}{4}\|x\|^2 + \frac{1}{4}\|u\|$$

$$\leq -(\frac{\sigma^{1-\alpha}}{RC} - \frac{1}{4})\|x\|^2 + \frac{1}{4}\|u\|$$

Letting $p = \frac{\sigma^{1-\alpha}}{RC} - \frac{1}{4}$ and $\theta \in (0,1)$, we have the following

$$D_\alpha^c V(t,x) \leq -(1-\theta)p\|x\|^2 + p\|x\|^2 + \frac{1}{4}\|u\| \quad (5.2)$$

If $\|x\| \geq \frac{\|x\|}{2p\theta}$, equation (5.2) reduces to

$$D_\alpha^c V(t,x) \leq -(1-\theta)p\|x\|^2$$

Therefore, the RC electrical circuit system (5.1) is Mittag-Leffler input stable. If the input $u = 0$ in equation (5.1), the electrical RC circuit system is Mittag-Leffler stable.

Example 2

Consider the system of fractional differential equations described by

$$D_\alpha^c x_1 = -3x_1 + \frac{1}{3}x_2 + \frac{1}{3}u_1$$

$$D_\alpha^c x_2 = -2x_2 + \frac{1}{3}u_2 \quad (5.3)$$

$$D_\alpha^c x_3 = -x_1 + \frac{1}{3}x_3 + \frac{1}{3}u_3$$

Where $x = (x_1, x_2, x_3) \in R^3$ and $u = (u_1, u_2, u_3) \in R^3$ represents the input. Let the Lyapunov function be defined by $V(x) = \frac{1}{3}(x_1^2 + x_2^2 + x_3^2)$. The Lyapunov function along the trajectories is given by

$$D_\alpha^c(t,x) = -3x_1^2 + \frac{1}{3}x_1x_2 + \frac{1}{3}x_1u_1 - 2x_2^2 + \frac{1}{3}x_2u_2 - x_3x_1 + \frac{1}{3}x_3^2 + \frac{1}{3}x_3u_3$$

$\leq -\frac{1}{3}x_1^2 - \frac{1}{3}x_2^2 - \frac{1}{3}x_3^2 + \frac{1}{9}\|u\|^2$
$\leq -V(x) + \frac{1}{9}\|u\|^2$

Where $\Omega_3(\|u\|) = \frac{1}{9}\|u\|^2$. According to theorem 3.1, the system of the fractional differential equations is Mittag-Leffler input stable.

If $u = 0$, equation (5.3) reduces to

$$D_\alpha^c x_1 = -3x_1 + \frac{1}{3}x_2$$

$$D_\alpha^c x_2 = -2x_2 \quad (5.4)$$

$$D_\alpha^c x_3 = -x_1 + \frac{1}{3}x_3$$

Where $x = (x_1, x_2, x_3) \in R^3$. Equation (5.4) is obviously Mittag-Leffler stable.

Example 3
Consider the fractional-order differential LC electrical system defined by

$$D_\alpha^c x = -\frac{1}{\sqrt{LC}}x + u \quad (5.5)$$

Where C denotes the capacitance, L represents the inductance, and x measures the intensity across the inductor. The solution of equation (5.5) is given by

$$x(t) = hE_\alpha(-\frac{1}{\sqrt{LC}}\left(\frac{t^k}{k}\right)^\alpha) + \int_0^1 (\frac{s^k - t^k}{k})^{\alpha-1} E_{\alpha,\alpha}(-\frac{1}{\sqrt{LC}}\left(\frac{t^k}{k}\right)^\alpha u(s)ds$$

Let $M = \int_0^1 (\frac{s^k - t^k}{k})^{\alpha-1} E_{\alpha,\alpha}(-\frac{1}{\sqrt{LC}}\left(\frac{t^k}{k}\right)^\alpha u(s)ds$

Applying the norm, we have

$$\|x(t)\| \leq \|h\| \left\| E_\alpha\left(-\frac{1}{\sqrt{LC}}\left(\frac{t^k}{k}\right)^\alpha\right)\right\| + \|u\|\|M\| \quad (5.6)$$

Equation (5.6) is in the form of (3.5).

If $\|h\| \left\| E_\alpha\left(-\frac{1}{\sqrt{LC}}\left(\frac{t^k}{k}\right)^\alpha\right)\right\| = \mu(\|x_0\|, t - t_0) =$
$\left\{\|h\| \left\| E_\alpha\left(-\frac{1}{\sqrt{LC}}\left(\frac{t^k}{k}\right)^\alpha\right)\right\|\right\}^{\frac{1}{q}}$ and

$\gamma(\|u\|_\infty) = \|u\|\|M\|$

With BIBS and CICS duly satisfied, equation (5.5) is Mittag-Leffler input stable. If $u = 0$, equation (5.5) is Mittag-Leffler stable.

4. Conclusion

Stability analysis of solutions of fractional differential equations is still occupying the interest of researchers due to its importance. Mittag-Leffler input stability as a special case of fractional input stability has been investigated in various fractional differential equations. It has been shown that when the input is zero, the system of fractional differential equations is generally Mittag-Leffler stable. The nature of the Mittag-Leffler input stability either in general or not in general is a function of the input. Examples have been given to illustrate the Mittag-Leffler stability analysis of fractional differential equations with and without input.

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