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A brief note on application of incline algebra in probable reasoning choice and automata theory

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Abstract
In This paper we make a brief note on application of incline algebra concepts and properties in probable reasoning choice and automata theory and many other field of mathematical science where we can use incline algebra concepts.

Keywords: Incline algebra, module, Pareto property, Permutation matrix, Probable Reasoning Choice, Automata theory.

Introduction

In our real life various problems related with medical sciences, engineering, political, financial, social deciplines and numerous different arenas involve provisional data which are not always necessarily in crisp, appropriate and conclusive forms due to uncertainty associated with these problems. Such problems are usually being handled with the help of the topics like probability theory, fuzzy set theory, intuitionistic fuzzy sets, interval mathematics and rough sets

Application of incline algebra in different areas are very active field of research since last 20 years. Incline algebra is generalization of both Boolean and fuzzy algebra and it is a special type of semi-ring which follows both a semi-ring structre and a poset structure. We take the basic notion and definitions from book authored Cao, Kim and Rough [1].

2. Prelimnriries

Definition 2.1. An incline is an algebraic structure $(\Im,+,*)$ having a non-empty set $\Im$ and two binary operations + and * such that for all x,y,z in $\Im$, if the following laws hold

[K1] *Associative laws*
1. $x + (y + z) = (x + y) + z$,
2. $x * (y * z) = (x * y) * z$.

[K2] *Commutative laws*
1. $x + y = y + x$,
2. $x * y = y * x$.

[K3] *Distributive laws*
1. $x * (y + z) = (x * y) + (x * z)$,
2. $(y + z) * x = (y * x) + (z * x)$.

[K4] *Idempotent law:* $x + x = x$.

[K5] *Incline law*
1. $x + (x * y) = x$,
2. $y + (x * y) = y$.

We shall also use the following definitions and properties in our further investigations

Definition 2.2. Let $x,y \in \Im$. The incline order relation denoted as "$\leq$" and is defined as $x \leq y \leftrightarrow x + y = y$.

From the incline axiom (K5) obiviously, we have

1. $x + y \geq x$ and $x + y \geq y$ for $x,y \in \Im$,
2. $xy \leq x$ and $xy \leq y$ for $x,y \in \Im$.

Which are known as incline properties.

Definition 2.3. A module over $\Im$ is a commutative semi-group M provided with a binary operation $\Im \times M \to M$ satisfying,

1. $a(x + y) = ax + ay$
2. $a(bx) = (ab)x,$
3. $(a + b)x = ax + bx$
4. $0 + x = x$ and
5. $1x = x$

- **Example 2.3.** A module over $\Im$ Let K be an ideal in $\Im$. Then K is a module over $\Im$**.**
- **Definition 2.4.** A free module over $\Im$ is a module isomorphic to a direct sum of Xerox of $\Im$.
- **Example 2.5.** The vector space v_n of all n −tuples of elements of $\Im$ is a free module. A free module is itself an incline.
- **Proposition 2.6.** A module over $\Im$ is a semi-lattice. We have

$ax + x = x, a\,0 = 0,$ and if $x + y = 0$ then $x = y = 0$

Proof:

1. $x + x = (1 + 1)x = 1x = x$
2. $x + ax = (1 + a)x = 1x = x$
3. $a0 = a0 + 0 = (a + 1)0 = (1)0 = 0.$

The last result is true in any semi-lattice.

Proposition 2.7. A free module over integral $\Im$ has a unique basis.

Proof: If there were two bases, the matrices A and B expressing each in terms of the other would satisfy $AB = BA = I$ By the previous theorem about invertible matrices over an integral incline A, B are permutation matrices.

The degree of a mapping from one free module to another is its algebraic degree in terms of coordinates: a linear map has degree 1, a bilinear or quadratic map has degree 2.

Definition 2.8. A finite state machine $(\mathcal{M}, X, v, \mathfrak{z})$ over $\Im$ consists of modules $\mathcal{M}, X, Z$ over $\Im$ and mappings $v: \mathcal{M} \times X \to \mathcal{M}$ and $\mathfrak{z}: \mathcal{M} \times X \to Z$.

Here $\mathcal{M}$ is the set of states, X is the set of inputs, Z is the set of outputs, $v(m, x)$ is the next state after a given state s and input x and $\mathfrak{z}\,(m, x)$ is the output from state s and input x. Its degree is sup (degree v, degree $\mathfrak{z}$).

Example 2.9. Every finite state machine yields a finite state machine over $\Im$ if $\Im$ has $0, 1$, where we take $\mathcal{M}, X, Z$ to be free modules and $v, \mathfrak{z}$ transformation matrices.

Example 2.10. Let $\Im = R^{+} \cup \{e\}$ where e is an identity element and let $\Im = \mathcal{M}, X, Z$ and let $v(m, x) = m + x = \mathfrak{z}(x)$. Then we have a machine which can count or add.

Group choice theory is concerned with the problem of evaluating m alternatives by a group of n individuals. Let X be the set of alternatives. In the simplest case, each individual has a preference relation on X, expressing the pairs (x, y) such that he prefers x to y.

This will be complete and transitive (a weak order). It will be a linear order if he values no two elements of X exactly the same. Then a group choice method gives a function from n −tuples of linear orders on X to some binary order on X. We write X as $\{x_1, x_2, \dots \dots \dots, x_n\}$.

Let $P < i >$ be the matrix of person $i's$ preference order.

Definition 2.11. A social welfare function with values in $\Im$ is a function F from L^n to $M_m(\Im)$ where L is the set of $m \times m$ matrices of linear.

Example 2.12. If we let F be the matrix i, j such that $F_{ij} = 1$ if and only if $|\{k: P < k >_{ij} = 1\}| > \frac{n}{2}$ then we have majority voting.

We will assume that the main diagonal entries of (i, i) are identically 1. Let F_{ij} denote the (i, j) − entry of F.

Definition 2.13. That F is independent of irrelevant alternatives means if $P < k >_{ij} = R < k >_{ij}$ and $P < k >_{ji} = R < k >_{ji}$ then, $F_{ij}(P_1, \dots \dots, P_n) = F_{ij}\,(R_1, \dots \dots, R_n)$.

Proposition 2.14. Here F is independent of irrelevant alternatives if and only if F_{ij} is a function of $R < k >_{ij}$.
Proof: For linear orders $R < k >_{ij} = R < k >_{ji}$ so knowledge of
$R < k >_{ij}$ is equivalent $R < k >_{ji}$ and $R < k >_{ji}$ The definition is equivalent to saying that is a partial function of $R < k >_{ij}$ and $R < k >_{ji}$ But on $R < k >_{ij}$ we have a function since the is all n −tuples of $\{0, 1\}$.

Definition 2.15. That F is neutral means for any permutation matrix $P, F(PR < 1 > P^T .., PR < n > P^T) = PF\ (R < 1 >, \ldots, R < n >)P^T$ That F is anonymous means for any permutation, $F(R < \pi(1) > \ldots, R < \pi(n) >) = F(R < 1 >, R < n >)$. More generally if this holds for a group G of permutations we say F is G −invariant.

Example 2.16. If F is a constant nonsymmetric relation then it is anonymous but not neutral.
Anonymity is symmetry in the persons, while neutrality is symmetry in the alternatives.

Definition 2.17. A social welfare function is transitive if and only if $F^2 \leq F$. It is *Pareto* if and $F = R < n >$.
This is a weak form of the *Pareto* property saying that if all individuals have identical preferences, the group preference must be the same as each individual's preference.

Example 2.18. Majority rule has the *Pareto* property. Its transitive closure Sigma $\sum F^n$ is transitive (but no longer independent of irrelevant alternatives).
Example 2.19. A constant function cannot have the *Pareto* property.

Proposition 2.20. If F is independent of irrelevant alternatives then it is also *Pareto* if and only if $F_{ij}(0, \ldots, 0) = 0$ and $F_{ij}\ (1, \ldots, 1) = 1$.

Proof: The Pareto condition in this case is equivalent to saying that if all $R < k >_{ij} = 0$ and all $R < k >_{ji} = 1$ then $F_{ij} = 0$ and $F_{ji} = 1$
This proves sufficiency. But $R < k >_{ij} = 0$ then $R < k >_{ji} = 1$ so $F_{ij} = 0$ and if $R < k >_{ij} = 1$ then $R < k >_{ji} = 0$ so $F_{ji} = 1$.

Proposition 2.21. Social welfare function to $\mathfrak{I}$ which are transitive, Pareto, independent of irrelevant alternatives, are in one-one correspondence with functions f: $V_n \rightarrow \mathfrak{I}$ such that (i) $f\ (0) = 0$, (ii) $f\ (1) = 1$, (iii) if $v \leq w$ then $f\ (v) \leq f(w)$, and (iv) $f\ (vw) \geq f(v)\, f(w)$, where $F_{ij}\ (R < 1 >_{ij}, \ldots \ldots \ldots, R < n >_{ij} = f(R < 1 >_{ij}, \ldots \ldots \ldots \ldots, R < n >_{ij}$ if there are at least three alternatives.

Definition 2.22. A social welfare function is oligarchical (dictatorial) if and only if there exists a set S(a single person) such that for
$(R < 1 >, \ldots \ldots \ldots .., R < n >)$, $F_{ij}\ (R < 1 >, \ldots \ldots \ldots .., R < n >) = 1$ if $x_i\ R < k > x_k\ \forall\ k \in S$ and $F_{ij} = 0$ otherwise.

Example 2.23. $F(R < 1 >, \ldots \ldots \ldots .., R < n >) = R < n >$ then we have a dictatorial social welfare function.

Corollary 2.24. For $\mathfrak{I} = \beta$, all social welfare functions satisfying the conditions of the theorem are oligarchical.

Corollary 2.25. For $\mathfrak{I} = \beta$, a social welfare function is complete and satisfies these conditions iff it is dictatorial.
In an oligarchical social welfare function, the group prefers x_i to x_j if and only if every member of the oligarchy does. Members outside the oligarchy have no effect on group decisions.

Proposition 2.26. Social welfare functions into $\mathfrak{I}$ which are anonymous, independent of irrelevant alternatives, Pareto, and transitive are in one-one Correspondence with functions g: $\{0\} \cup n \rightarrow \mathfrak{I}$ such that
$g(0) = 0, g(1) = 1, g(n - a - b) \geq g(n - a)\, g(n - b)$ and if $a \leq b$ then $g(a) \leq g(b)$.

Proof: That a social welfare function is anonymous means $f(v)$ depends only on the number of ones in v. Let $g(a)$ be $f(v)$ where v has a ones. Then the conditions of the last theorem translate directly to those here.

Proposition 2.27. Suppose $\mathfrak{I}$ has no nilpotent elements Then every Pareto, anonymous, and transitive social welfare function is a multiple of the unanimity function in which the group prefers i to j if and only if all individuals do.
Therefore, this holds for $\mathfrak{I}_1$ and $\mathfrak{I}_3$. However for $\mathfrak{I}_2$ there does exist a social welfare function which precisely represents majority rule. Here $0, 1$ are switched.

Proposition 2.31. A logical formula can be expressed solely in terms $\rightarrow$ if and only if it has the form $p_1 \vee z$ for some basic variable z and formula p_1.

1. **Proof:** From $y \rightarrow z$ we have $\sim y \vee z$ for all z.
2. If $p_1 \vee z$ is obtained then $(p_1 \vee z) \rightarrow z$ gives $(\sim p_1 \wedge \sim z) \vee z = \sim p_1 \vee z$.
3. If $p_1 \vee z$ and $p_2 \vee z$ are obtained we obtain $(p_1 \vee z) \rightarrow (p_2 \vee z)$ or $(\sim p_1 \wedge \sim z) \vee p_2 \vee z = \sim p_1 \vee p_2 \vee z$.

So the class of p_1 which can be obtained contains $\sim y$ for all variables and is closed under negation and or. So it consists of all formulas p_1. We have a formula $p_1 \rightarrow p_2$ then we have $\sim p_1 \vee p_2$. By induction we may suppose p_2 has the form $p_3 \vee z$. So all obtainable formulas have this form.
The operation $\vee$ can be defined solely in terms by

$$x \vee y = (x \rightarrow y) \rightarrow \text{ y}.$$

Therefore, if $\rightarrow$ is defined in a structure, it must be a semi-lattice under $\vee$.

Proposition 2.32. The element $z = x \rightarrow y$ in any Boolean algebra is uniquely characterized by (i) $z \geq y$ (ii) $z \vee x = 1$, and (iii) $z \wedge (x \vee y) = y$.
These equations can define operations with some of the properties of $\rightarrow$ (if then).
However, it does not exist in most inclines. For example, if the incline is linearly ordered, and $0 < x < 1$ and $x \nleq y$, then (ii) implies $z = 1$ but this contradicts (iii) of Proposition 5.2.32.

Example 2.28. For $\mathfrak{I}_2$ let F_{ij} be $\frac{k}{n}$ where k is the number of voters who do not prefer i to j.
This gives a social welfare function which is transitive, Pareto optimal, neutral, and independent of irrelevant alternatives.

3. Application of Incline Algebra in Probable Reasoning Choice and Automata theory

Inclines can be used in decision theory. Let $A = (a_{ij})$ be the matrix of a decision table, that is a_{ij} is the value of choice i under state j of nature. Then a decision rule is to maximize $f(a_{i1}, a_{i2}, \dots \dots, a_{in})$ where f is some function. For the maxmin rule f is infimum. Then we find $\sum \prod a_{i1}, a_{i2}, \dots \dots, a_{in}$ for the incline $\mathfrak{I}_1$ (or its dual). For the Nash bargaining solution (assigning 0 as disagreement value) we use the same formula for the incline $\mathfrak{I}_3$. This maximizes the ordinary product $a_1 a_2 \dots \dots \dots \dots a_n$. Other inclines give other choice rules.

Example 3.1. We can apply this to a choice between two alternatives a, b under three possibilities $1, 2, 3$:

1. a 0.4 0.7 0.8
2. b 0.5 0.5 0.5

Then under the $\mathfrak{I}_1$ choice rule we have $sup\{ 0.4, 0.5 \} = 0.5$ so $b$is chosen. But under the $\mathfrak{I}_3$ rule we have sup $\{(0.4)\ (0.7)\ (0.8), ((0.5)^3 = (0.4)(0.7)(0.8)$ so a is chosen.
These rules can be applied to selection of an individual for promotion.

Example 3.2. Consider an individual's score on three attributes: human relationships, job performance, education. Let these be $0.2, 0.3, 0.8$ for individual a; $0.3, 0.3, 0.7$ for individual b; $0.6, 0.6, 0.6$ for individual c. Then by the $\mathfrak{I}_3$ choice rule we consider the supremum of the products $0.048, 0.063, 0.216$ and so the last individual is chosen.
Ma and Cao (1982a, 1982b, 1983) give applications of inclines to multistage evaluation in psychological measurement and decision-making.
Logic is an algebra with operators and, or, if then, not, if and only if. The operations and, or are structurally analogous to sum, product in inclines. What about if then? We can consider logic as a structure solely in this operation.
Firstly method to use inclines in probable reasoning suppose each of a set of statement i supports by some conclusion C.let if statement i supports conclusion C to a degree a_i, and one's supports in statement i is p_i. then one's supports in conclusion C can be taken as at least $sup\{a_i\ p_i\}$.
If in fact all data were i known to be independent a higher value could be obtained by

$$1 - (1 - a_1 p_1)\ (1 - a_2 p_2) \dots \dots \dots \dots \dots (1 - a_n p_n).$$

However for most values the lower value $sup\{a_i\, p_i\}$ is a good approximate value.

Proposition 3.3: if x, y are independent and uniformly distributed in [0,1] the approximated value of $1 - (1 - x)\ (1 - y) - sup\{x, y\}$ is $1/12$.

Proof: Here we have

$$2 \iint_{\mathbf{0}}^{y} \{1 - (1 - x)(1 - y) - y\}\, dx\, dy = 1/12$$

In general, sup { Prob (p_i) } can be used to estimate of the probability of $p_1 \vee p_2 \vee \dots \dots . \vee p_n$. and it is a lower bound and will be exact if and only if some p_i implies rest of p_i.
If it is desired to have the quantity x_i as well as the utility enter the calculation, we may use a product structure. Then $sup\ \{(a_i x_i, x_i)\}$ represents the utility and quantity chosen where we effectively order the pairs according to the first factor only. This can be achieved, for example by a lexicographic linear order or a partial order.
We can consider finite machines over an incline $\mathfrak{I}$ in basically the same way as linear systems.

A cybernetic mechanism can govern a process by optimizing some quantity which is its goal. For instance, it can minimize the distance from an ideal state. If the quantity can be expressed as a for choice of i then again we have $sup\ (a_i x_i)$ Inclines can be used to represent binary relations in which there is a notion of degree as well as existence or nonexistence of the relationship, and in which the composition may have lesser degree than its factors.

Furthermore, inclines can be used to represent choice or decision behavior by a consumer or a mechanism governing some process. For example, a consumer is assumed to choose the bundle of goods giving him the highest utility among all those he can purchase. Therefore, if he can afford amount x_i of good i and the utility of this to him is a_i per unit, his utility is $sup\ (a_i x_i)$ over i. Inclines can be used to indicate quality as opposed to quantity, or order of magnitude, since sums preserve values. The sum of two items of quality x may be taken as quality x.

4. Conclusion

In this paper, we defined some new concept and properties of incline algebra and use this to application in probable reasoning choice and automata theory. Also gathered application of incline algebra in different fields.

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