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Clique and distance analysis in the undirected power graph of Z_n

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Abstract
In this paper, we investigate the structural properties of the undirected power graph $G(Z_n)$ associated with the finite cyclic group Z_n. We begin by establishing the necessary and sufficient conditions under which $G(Z_n)$ It is bipartite and characterizes its clique structure. Explicit formulas are derived for the degree of a vertex based on the subgroup structure of Z_n, leading to a criterion for identifying the minimum degree in the graph. We also analyze distance-related parameters, including radius, diameter, and average distance, and explore conditions under which these attain extremal values. The number of vertex pairs at distance two is computed for specific group orders, such as $G(Z_{pq})$ and $G(Z_{p^2q})$, revealing intricate combinatorial relationships. These results enhance the understanding of how algebraic properties of Z_n influence the topology of its power graph.

Keywords: Power graph, cyclic group, distance in graphs, clique, degree, Euler's Totient function

Introduction

Graphs arising from algebraic structures have received considerable attention due to their ability to capture intrinsic relationships within groups, rings, and semigroups. One such construction is the power graph of a group, introduced as a way to represent the power relations among elements of a group in a graphical framework. Given a group G, its (undirected) power graph $G(G)$ is defined as the simple graph with the vertex set $G(G)$, where two distinct elements x and y are adjacent if one is a power of the other.

In this paper, we focus on the undirected power graph of the finite cyclic group Z_n, denoted $G(Z_n)$. As every finite cyclic group of order n is isomorphic to Z_n, the study of $G(Z_n)$ provides a complete understanding of the power graph structure for all finite cyclic groups. This graph encapsulates not only the algebraic hierarchy of the group but also exhibits interesting combinatorial and metric properties.

We explore several structural aspects of $G(Z_n)$, including conditions for bipartiteness, identification of cliques, and determination of vertex degrees based on subgroup information. Furthermore, we examine metric parameters such as diameter, radius, and average distance between vertices. Particular attention is paid to the distribution of vertex pairs at distance two, which reveals deep connections between group-theoretic properties and graph distances.

The results presented here not only extend earlier work on power graphs of cyclic groups but also offer new insights into how the subgroup lattice and Euler's totient function influence the topology of the associated graph. Our findings highlight the intricate interplay between algebra and graph theory and contribute to the growing field of algebraic graph theory.

Undirected Power Graph of Z_n

$G(Z_n)$ is the undirected power graph of Z_n or the underlying simple graph of the directed power graph of Z_n.

Theorem 2.1. *$G(Z_n)$ is bipartite if and only if $n=2$ or $n=1$.*

Proof: $G(Z_1)$ and $G(Z_2)$ are trivially bipartite. Suppose $n>2$. If possible, assume that $G(Z_n)$ is bipartite with a bipartition (X,Y). Since 0 is adjacent to all other vertices in $G(Z_n)$, either X or Y must be a singleton set containing 0. Let $X=\{0\}$. Now Y contains all other vertices of $G(Z_n)$. Since $n\geq 3$, Z_n has at least two generators, and the vertices corresponding to these generators are adjacent in Y, which is impossible. Hence $G(Z_n)$ is not bipartite.

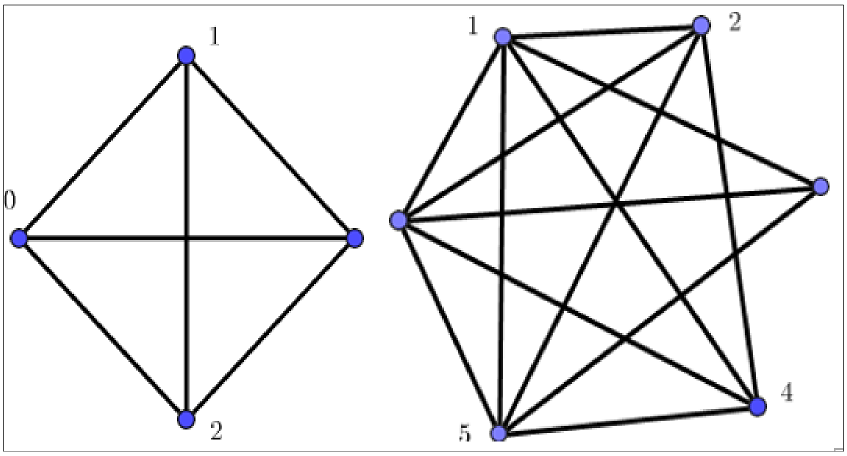

Fig 1: $G(Z_4)$ and $G(Z_6)$

Remark 2.2. $G(Z_n)$ contains a clique of order $\phi(n)+1$, since the vertices corresponding to the $\phi(n)$ generators of Z_n and the vertex 0 is adjacent to each other in $G(Z_n)$.

Theorem 2.3 Let $0\neq b$ be an element of Z_n having order m and let H be the cyclic subgroup generated by b. Let H_i be the subgroups of Z_n having order m_i where m divides m_i. Then $\deg(b)$ is $\phi(n)+\phi(m_1)+\phi(m_2)+.....+\phi(m_r)+m-1$ in $G(Z_n)$.

Theorem 2.4 In $G(Z_n)$ the minimum degree $\delta(G(Z_n))\geq\phi(n)+1$ and $\delta(G(Z_n))=\phi(n)+1$ if and only if $n=2k$, for some odd number k

Proof: Here the vertices corresponding to the $\phi(n)$ generators of Z_n and the vertex 0 have degree $n-1$. Since these $\phi(n)+1$ vertices are adjacent to all other vertices of $G(Z_n)$, the minimum degree $\delta(G(Z_n))\geq\phi(n)+1$.

Now suppose $n=2k$, where k is odd. Let $b\in V(G(Z_n))$ have order 2. Since k is odd, there does not exist a subgroup of Z_n having order m_i where 2 divides m_i.

Hence by Theorem 2.3 degree of b in $G(Z_n)$ is,

$$\deg(b)=\phi(n)+2-1=\phi(n)+1$$

Conversely, suppose that $\delta(G(Z_n))=\phi(n)+1$. Then there exists a non-zero vertex b having order m in $G(Z_n)$ such that $\deg(b)=\phi(n)+1$. We have by Theorem 2.3,

$\deg(b)=\phi(m_1)+\phi(m_2)+.....+\phi(m_r)+m-1$ where m_i, $i=1,2,...,r$ are the orders of all subgroups H_i of the group Z_n, where m divides m_i.

That is,

$$\phi(m_1)+\phi(m_2)+.....+\phi(m_r)+m-1=\phi(n)+1$$

Hence,

$$\phi(m_1)+\phi(m_2)+.....+\phi(m_r)+m=2$$

This is possible only if $m=2$, and $\phi(m_i)=0, \forall i=1,2,...,r$. Then there does not exist H_i such that $<b> \le H \le Z_n$. Since m divides $n, n=2k$, for some k. If k is even, then $n=4t$, for some t. Then there exists a subgroup H of Z_n such that $|H|=2t$ and $<b> \le H \le Z_n$, a contradiction. Hence k must be an odd number.

In $G(Z_n)$ all the vertices are either at a distance 1 or 2. Moreover, 0 and the generators of Z_n are at a distance of 1 from all the vertices of $G(Z_n)$. Hence the radius of $G(Z_n)$ is $rad(G(Z_n))=1$ and the diameter of $G(Z_n)$ is $diam(G(Z_n))=1$ if and only if $n=p^r$, for some prime p and is $diam(G(Z_n))=2$ otherwise. Hence the average distance $\mu(G(Z_n))$ between vertices of $G(Z_n)$ is in [1, 2]. That is, $1 \le \mu(G(Z_n)) \le 2$ and $\mu(G(Z_n))$ and m, the number of edges of $G(Z_n)$ are inversely proportional. Also, $\mu(G(\square_n))=1$ if and only if $n=p^r$, for some prime p.

Remark 2.5: In $G(Z_n)$ the distance *degree* of any vertex v is $2(n-1-\deg(v))+\deg(v)$.

Proof*:* Since $\deg(v)$ vertices are adjacent to v and $n-1-\deg(v)$ vertices are non- adjacent to v, these $n-1-\deg(v)$ vertices are at a distance of 2 from v. Hence the result.

Remark 2.6: The number of vertex pairs at distance 2 in $G(Z_{pq})$) is $\phi(p)\phi(q)$.

Proof: From the component digraph in figure 2 of $G(Z_{pq})$ the $\phi(p)$ vertices of degree q and $\phi(q)$ vertices of degree p are nonadjacent. Therefore $\phi(p)\phi(q)$ is the number of vertex pairs of $G(Z_{pq})$ at distance 2. Hence, we get the result.

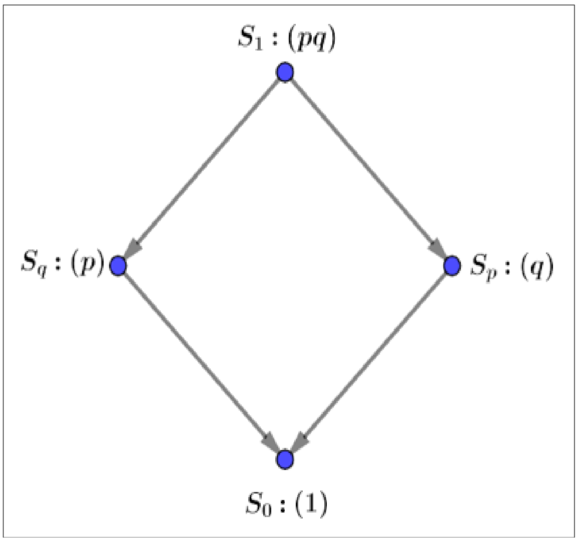

Fig 2: Component digraph of $G(Z_{pq})$.

Remark 2.7: The number of vertex pairs at distance 2 in $G(Z_{p^2q})$ is $\phi(p)[\phi(pq)+\phi(q)]+\phi(p)\phi(q)$.

Proof: From the component digraph 3 of $G(Z_{p^2q})$ the $\phi(pq)$ elements of S_p and $\phi(p^2)$ elements of S_q are nonadjacent. Similarly, the $\phi(q)$ elements of S_{p^2} are nonadjacent to $\phi(p^2)$ elements of S_q and $\phi(p)$ elements of S_{pq}. Hence, the number of vertex pairs at a distance 2 in $G(Z_{p^2q})$ is $\phi(p)[\phi(pq)+\phi(q)]+\phi(p)\phi(q)$.

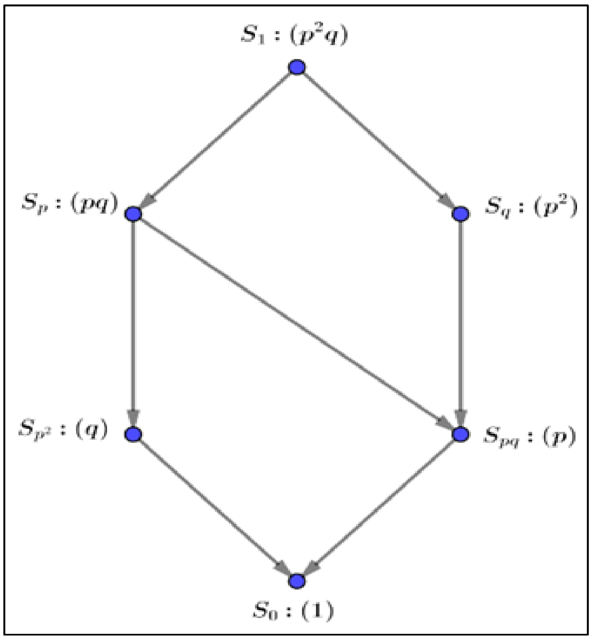

Fig 3: Component digraph of $G(Z_{p^2q})$

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