International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2025; 10(8): 123-128
© 2025 Stats & Maths
https://www.mathsjournal.com
Received: 12-07-2025
Accepted: 09-08-2025

Saraswati V Katagi
Department of Studies in Statistics, Karnatak University, Dharwad, Karnataka, India

SB Munoli
Department of Studies in Statistics, Karnatak University, Dharwad, Karnataka, India

Corresponding Author:
Saraswati V Katagi
Department of Studies in Statistics, Karnatak University, Dharwad, Karnataka, India

Modelling and evaluation of 1-out-of-3 system reliability using extension of bivariate Freund distribution

Saraswati V Katagi and SB Munoli

DOI: https://www.doi.org/10.22271/maths.2025.v10.i8b.2133

Abstract
This study analyzes the reliability of a 1-out-of-3 system using the Freund bivariate exponential model to account for dependent component failures. The mean residual life (MRL), hazard rate and reliability function expressions are derived and analyzed under various failure rate scenarios. Maximum likelihood estimation (MLE) is employed for reliability estimation and a simulation study confirms the consistency of the estimators. The model effectively captures dynamics of reliability, failure rate, MRL due to failure of components in 1-out-of-3 system making it suitable for applications in mechanical and computational systems with redundant configurations.

Keywords: MLE, 1-out-of-3 system, Freund model, reliability characteristics, mean absolute bias

1. Introduction

The dependability of a system is determined not solely by the reliability of its constituent components but also by the configuration of the system, the interdependence among components and the characteristics of their failure time distributions. Among the numerous configurations, the s-out-of-k system $(s \leq k)$ serves a pivotal role in augmenting system reliability, particularly within the realm of reliability engineering applications. This model serves to generalize the two fundamental types of systems: the parallel system (when $s = 1$) and the series system (when $s = k$). Consequently, the s-out-of-k configuration offers a versatile framework that reconciles the dichotomies of redundancy and susceptibility in the design of systems. Because, of this flexibility it is widely suitable in various practical scenarios, especially in mission-critical and fault-tolerant applications such as aerospace, communication networks, power systems and cloud computing where redundancy is a key design feature. But it is also true that the workload or pressure on the functioning components will change due to failure of components in the system. So, there will be change in the hazard rate of the functioning components. For instance, in cloud computing environments, data centers often rely on redundant server clusters to ensure high availability and fault tolerance. When a server within a cluster fails, the computational workload and network traffic are redistributed among the remaining active servers. This redistribution of load can lead to increased Central Processing Unit (CPU) utilization, memory consumption and thermal stress, which in turn elevates the failure probability of the surviving servers. Such stress-induced dependence among server lifetimes necessitates the use of models like the Freund distribution, which explicitly accounts for failure rate transitions due to component interdependence. Thus, despite the presence of redundancy, the reliability of the system may deteriorate more rapidly following failures.

Freund (1961) [1] has proposed a bivariate exponential model capturing dependent lifetimes and has applied it to real-world systems like engine failures and paired organs. Extensive work has been done using Frend bivariate exponential distribution, and few of them are: Boardman (1968) [2] has developed MLE methods for mixed exponential distributions in life testing.

Block and Basu (1974) [3] have introduced the ACBVE model to generalize Freund's distribution with continuous dependency, provided estimation methods and highlighted its practical relevance. Ross (1984) [4] has modeled failure rates that adjust based on the current working set of components, derived system failure distributions and explored repair-based reversibility. Hanagal (1992) [5] has extended Freund's model with large-sample inference for independence and symmetry, confidence intervals under censoring schemes are also obtained. Kunchur and Munoli (1994) [6] have estimated reliability in Freund-based two-component systems. Kim and Kvam (2004) [7] have developed MLEs for systems with unknown load-sharing rules. Kvam and Peña (2005) [8] have proposed a semi-parametric estimator for dynamic load-sharing systems. Park (2010) [9] has derived MLEs and BUEs under exponential and Weibull load-sharing systems. Gurler *et al.* (2015) [10] have studied mean remaining strength in dependent-component systems under stress. Asha *et al.* (2018) [11] have modeled load-sharing systems with frailty using bivariate Weibull and positive stable distributions. Zhao *et al.* (2018) [12] have built a continuous degradation model for load-sharing systems using log-linear stress-degradation links. Katagi and Munoli (2025) [13] have assessed 2-out-of-3 system reliability using Freund's model with MLE and Bayesian methods. Kundu (2025) [14] has proposed a bivariate load-sharing model for dependent failures with ties.

In the present study 1-out-of-3 system, which continues to function until the failure of its last component is considered and analyzed in the context of Freund model. The model formulation is detailed in Section 2. Section 3 provides an analysis of the reliability function, hazard rate and survival function. Parameter estimation based on life testing and simulation studies are presented in Sections 4 and 5, respectively. The concluding remarks are outlined in Section 6.

2. Model Description

Consider a system of three components arranged in a 1-out-of-3 configuration, that is the system functions as long as at least one component is operational. In such a configuration, as individual components experience failure sequentially, the parameter of the life distribution of the remaining operational components will change (without any form of replacement or repair). Specifically, let the lifetimes of the system's components be independent exponential random variables with a common failure rate parameter λ. Upon the failure of any one of the three components, the failure rate of the two remaining components changes to λ_1. Once the second failure occurs, the failure rate of the single surviving component changes to λ_2.

Depending on the application, the relationship among the failure rates may vary. For example, the condition $\lambda < \lambda_1 < \lambda_2$ is typically observed in reliability engineering, where increased stress accelerates component degradation. In contrast, the condition $\lambda > \lambda_1 > \lambda_2$ may arise in ecological or competing species scenarios, where the failure (extinction) of one species reduces competition and thereby enhances the survival of others.

3. Reliability function, hazard rate and mean residual life

3.1 Reliability Function

Reliability of 1-out-of-3 system with above property of changing life distribution upon subsequent failures at time t is obtained as below

$$R(t) = e^{-3\lambda t} + 3\int_0^t \lambda e^{-3\lambda t_1}\, e^{-2\lambda_1(t-t_1)}\, dt_1 + 3\int_0^t\int_{t_1}^t \lambda e^{-3\lambda t_1}\,\lambda_1\, e^{-2\lambda_1(t_2-t_1)}\, e^{-\lambda_2(t-t_2)}\, dt_2\, dt_1 \quad (1)$$

The first term on the right-hand side corresponds to the event that all three components of the system have survived throughout the interval $(0, t)$. The second term represents the scenario in which two out of the three components survive until time t, where the first failure occurs at time t_1. The remaining two components operate with failure rate λ up to time t_1, and subsequently with failure rate λ_1. The third term accounts for the case where only one component survives until time t, following both the first and second failures. The components initially function with rate λ, transition to rate λ_1 after the first failure at t_1, and finally last component operates under rate λ_2 after the second failure at t_2.

Representing (1) as

$$R(t) = e^{-3\lambda t} + I_1 + I_2 \quad (2)$$

with $I_1 = 3\int_0^t \lambda e^{-3\lambda t_1}\, e^{-2\lambda_1(t-t_1)}\, dt_1$

$$\Rightarrow I_1 = \frac{3\lambda\left(e^{-2\lambda_1 t} - e^{-3\lambda t}\right)}{(3\lambda - 2\lambda_1)} \quad (3)$$

$$I_2 = 3\int_0^t\int_{t_1}^t \lambda e^{-3\lambda t_1}\,\lambda_1\, e^{-2\lambda_1(t_2-t_1)}\, e^{-\lambda_2(t-t_2)}\, dt_2\, dt_1$$

$$\Rightarrow I_2 = \frac{3\lambda\lambda_1}{\lambda_2 - 2\lambda_1}\left\{\frac{\left(e^{-2\lambda_1 t} - e^{-3\lambda t}\right)}{(3\lambda - 2\lambda_1)} - \frac{\left(e^{-\lambda_2 t} - e^{-3\lambda t}\right)}{(3\lambda - \lambda_2)}\right\} \quad (4)$$

Substituting I_1 and I_2 of (3) and (4) into the Equation (2) and simplifying, the closed form expression for system reliability is obtained as

$$R(t) = e^{-3\lambda t} + \frac{3\lambda\left(e^{-2\lambda_1 t} - e^{-3\lambda t}\right)}{3\lambda - 2\lambda_1} + \frac{3\lambda\lambda_1}{\lambda_2 - 2\lambda_1}\left\{\frac{\left(e^{-2\lambda_1 t} - e^{-3\lambda t}\right)}{(3\lambda - 2\lambda_1)} - \frac{\left(e^{-\lambda_2 t} - e^{-3\lambda t}\right)}{(3\lambda - \lambda_2)}\right\} \quad (5)$$

$R(t)$ is well-defined, when $3\lambda \neq 2\lambda_1$, $\lambda_2 \neq 2\lambda_1$ and $3\lambda \neq \lambda_2$ and satisfies the boundary conditions $R(0) = 1$, $R(\infty) = 0$ with $R(t)$ being non-increasing in t.

3.2 Hazard Rate

Hazard rate is the probability that a unit that has functioned without failure up to the instant t will fail in the interval $(t, t + \Delta t)$ and is denoted by $h(t)$

$$h(t) = \frac{f(t)}{R(t)} \quad (6)$$

With $f(t)$ being probability density function (PDF) of the system lifetime, which is given by $-\frac{dR(t)}{dt}$ and is obtained as:

$$f(t) = 3\lambda e^{-(3\lambda t)} + \frac{3\lambda\left(2\lambda_1 e^{-2\lambda_1 t} - 3\lambda e^{-3\lambda t}\right)}{3\lambda - 2\lambda_1} - \frac{3\lambda\lambda_1}{\lambda_2 - 2\lambda_1}\left\{\frac{\left(3\lambda e^{-3\lambda t} - 2\lambda_1 e^{-2\lambda_1 t}\right)}{(3\lambda - 2\lambda_1)} - \frac{\left(3\lambda e^{-3\lambda t} - \lambda_2 e^{-\lambda_2 t}\right)}{(3\lambda - \lambda_2)}\right\} \quad (7)$$

which is valid under the same conditions imposed for Equation (5), ensuring that all denominators of Equation (7) are non-zero.

Hazard rate of the model is derived as:-

$$h(t) = \frac{3\lambda e^{-(3\lambda t)} + \frac{3\lambda\left(2\lambda_1 e^{-2\lambda_1 t} - 3\lambda e^{-3\lambda t}\right)}{3\lambda - 2\lambda_1} - \frac{3\lambda\lambda_1}{\lambda_2 - 2\lambda_1}\left\{\frac{\left(3\lambda e^{-3\lambda t} - 2\lambda_1 e^{-2\lambda_1 t}\right)}{(3\lambda - 2\lambda_1)} - \frac{\left(3\lambda e^{-3\lambda t} - \lambda_2 e^{-\lambda_2 t}\right)}{(3\lambda - \lambda_2)}\right\}}{e^{-3\lambda t} + \frac{3\lambda(e^{-2\lambda_1 t} - e^{-3\lambda t})}{3\lambda - 2\lambda_1} + \frac{3\lambda\lambda_1}{\lambda_2 - 2\lambda_1}\left\{\frac{\left(e^{-2\lambda_1 t} - e^{-3\lambda t}\right)}{(3\lambda - 2\lambda_1)} - \frac{\left(e^{-\lambda_2 t} - e^{-3\lambda t}\right)}{(3\lambda - \lambda_2)}\right\}} \quad (8)$$

The behavior of the $h(t)$ is analyzed under two distinct sets of parameter values, and the corresponding results are illustrated graphically in Figure 1.

3.3 Mean Residual Life

The mean residual life (MRL) at time t, denoted by $MRL(t)$, represents the expected remaining lifetime of the system, conditioned on survival up to time t. It is given by

$$MRL(t) = \frac{\int_t^\infty R(x)\,dx}{R(t)} \quad (9)$$

Let the integrated reliability function be $R^*(t) = \int_t^\infty R(x)\,dx$.

Using the reliability function $R(x)$ from Equation (5), the expression for $R^*(t)$ is obtained as follows:-

$$R^*(t) = \frac{e^{-3\lambda t}}{3\lambda} + \frac{3\lambda}{3\lambda - 2\lambda_1}\left(\frac{e^{-2\lambda_1 t}}{2\lambda_1} - \frac{e^{-3\lambda t}}{3\lambda}\right) + \frac{3\lambda\lambda_1}{\lambda_2 - 2\lambda_1}\left\{\frac{1}{3\lambda - 2\lambda_1}\left(\frac{e^{-2\lambda_1 t}}{2\lambda_1} - \frac{e^{-3\lambda t}}{3\lambda}\right) - \frac{1}{3\lambda - \lambda_2}\left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-3\lambda t}}{3\lambda}\right)\right\} \quad (10)$$

Thus, the MRL function reduces to:-

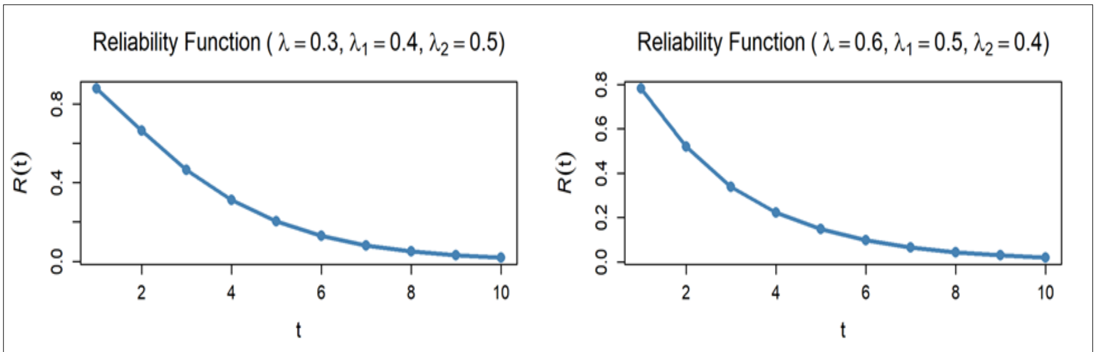
$$MRL(t) = \frac{\frac{e^{-3\lambda t}}{3\lambda} + \frac{3\lambda}{3\lambda - 2\lambda_1}\left(\frac{e^{-2\lambda_1 t}}{2\lambda_1} - \frac{e^{-3\lambda t}}{3\lambda}\right) + \frac{3\lambda\lambda_1}{\lambda_2 - 2\lambda_1}\left\{\frac{1}{3\lambda - 2\lambda_1}\left(\frac{e^{-2\lambda_1 t}}{2\lambda_1} - \frac{e^{-3\lambda t}}{3\lambda}\right) - \frac{1}{3\lambda - \lambda_2}\left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-3\lambda t}}{3\lambda}\right)\right\}}{e^{-3\lambda t} + \frac{3\lambda(e^{-2\lambda_1 t} - e^{-3\lambda t})}{3\lambda - 2\lambda_1} + \frac{3\lambda\lambda_1}{\lambda_2 - 2\lambda_1}\left\{\frac{\left(e^{-2\lambda_1 t} - e^{-3\lambda t}\right)}{(3\lambda - 2\lambda_1)} - \frac{\left(e^{-\lambda_2 t} - e^{-3\lambda t}\right)}{(3\lambda - \lambda_2)}\right\}} \quad (11)$$

The variation in MRL is analyzed for two distinct parameter sets and presented graphically in Figure 1.

Table 1: Theoretical $R(t)$, $h(t)$ and $MRL(t)$ values for increasing and decreasing failure rates

$Sl.no$	t	$\lambda = 0.3, \lambda_1 = 0.4, \lambda_2 = 0.5$			$\lambda = 0.6, \lambda_1 = 0.5, \lambda_2 = 0.4$		
		$R(t)$	$h(t)$	$MRL(t)$	$R(t)$	$h(t)$	$MRL(t)$
1	1	0.878175	0.224692	2.739710	0.782361	0.373747	2.418908
2	2	0.663248	0.325725	2.463379	0.519977	0.425719	2.399212
3	3	0.464442	0.382087	2.310861	0.339362	0.424691	2.429030
4	4	0.311043	0.417193	2.216778	0.222853	0.416398	2.456678
5	5	0.202417	0.440516	2.154807	0.147474	0.409757	2.475026
6	6	0.129189	0.456651	2.112256	0.098118	0.405554	2.485970
7	7	0.081335	0.468113	2.082227	0.065494	0.403101	2.492213
8	8	0.050709	0.476399	2.060638	0.043799	0.401715	2.495703
9	9	0.031391	0.482463	2.044918	0.029322	0.400945	2.497635
10	10	0.019331	0.486935	2.033371	0.019641	0.400520	2.498700

Table 1 depicts the behavior of $R(t)$, $h(t)$ and $MRL(t)$ functions at various time points for ascending and descending failure rates. In both the cases reliability values decrease as time progresses which is true in practical situations also, on the other hand the characteristics of hazard rate and mean residual life are different for two sets of failure rates. For ascending case of failure rates $h(t)$ increases in contrast $MRL(t)$ decreases over t values whereas, in the case of descending failure rates, the nature of $h(t)$ and $MRL(t)$ is bit different. In time interval $(1, 2)$ hazard rate is increasing later, decreasing trend is observed and exactly contrasting behavior has been shown by $MRL(t)$.



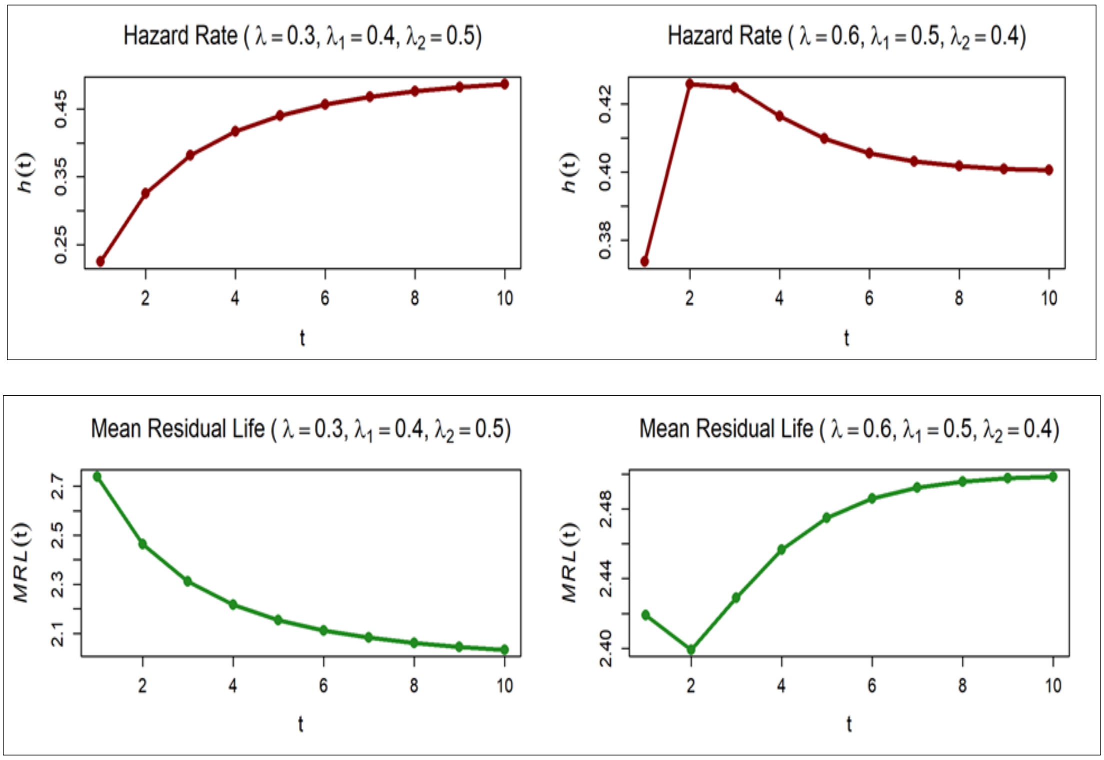


Fig 1: $R(t)$, $h(t)$ and $MRL(t)$ for increasing and decreasing failure rates

Figure 1, displays the trend of $R(t)$, $h(t)$ and $MRL(t)$ for two sets of failure rate parameters, representing increasing and decreasing failure rates. Reliability decreases monotonically over time in both the cases of failure rates, which is expected. This aligns with the system becomes more prone to failure as time elapses. The hazard rate at the starting period of time increases and eventually attains constant value corresponding to the hazard rate of the last surviving component in increasing failure rate scenario. Whereas, it raises steeply during the initial period of time and then gradually declines, ultimately approaching a stable value associated with the final component in decreasing failure rate situation. The Mean residual life curve of the system exhibits a nature that is reciprocal to the hazard rate in each case: it decreases over time in the increasing failure rate scenario and increases over time in the decreasing failure rate scenario.

4. Estimation of Parameters by Life Testing.

A system with the features described in Section 2 is set into life testing and is observed until all components have failed. Let U, V and W respectively denote the times at which the first, second and third failures occur.

- The probability that the first failure occurs at time $U = u$ and other two components remain operational is given by $\lambda e^{-\lambda u}\, e^{-\lambda u}\, e^{-\lambda u} = \lambda e^{-3\lambda u}$
- The probability that the second failure takes place at time $V = v$, with the one component still functioning, given that the first failure occurred at $U = u$ is given by $\lambda_1 e^{-\lambda_1(v-u)}\, e^{-\lambda_1(v-u)} = \lambda_1 e^{-2\lambda_1(v-u)}$
- The probability that the third failure is observed at time $W = w$, given that the first and second failures respectively occurred at times $U = u, V = v$ is given by $\lambda_2 e^{-\lambda_2(w-v)}$

The joint pdf of U, V and W is as follows

$$f_{U,V,W}(u, v, w) = 3!\ \lambda e^{-3\lambda u}\ \lambda_1 e^{-2\lambda_1(v-u)}\ \lambda_2 e^{-\lambda_2(w-v)},\ 0 < u < v < w < \infty. \quad (12)$$

For all values of λ, λ_1 and λ_2 defined on the interval $(0, \infty)$ and $0 < u < v < w < \infty$, the Equation (12) is non-negative and satisfies the condition

$$\int_0^\infty \int_u^\infty \int_v^\infty f_{U,V,W}(u, v, w)\, dw\, dv\, du = 1.$$

When n such systems subjected to life test and joint likelihood function is given by

$$L(\lambda, \lambda_1, \lambda_2) = \prod_{i=1}^{n} f_{U,V,W}(u_i, v_i, w_i)$$

$$L(\lambda, \lambda_1, \lambda_2) = 6^n\ \lambda^n e^{-3\lambda \sum_{i=1}^{n} u_i}\ \lambda_1^n e^{-2\lambda_1 \sum_{i=1}^{n}(v_i - u_i)}\ \lambda_2^n e^{-\lambda_2 \sum_{i=1}^{n}(w_i - v_i)} \quad (13)$$

Using the natural logarithm function to both sides of the Equation (13), followed by computing the partial derivatives concerning each parameter, and equating these derivatives to zero, the maximum likelihood estimators (MLE) of $\lambda, \lambda_1, \lambda_2$ are:

$$\hat{\lambda} = \frac{1}{3\bar{u}},\ \hat{\lambda}_1 = \frac{1}{2(\bar{v} - \bar{u})},\ \hat{\lambda}_2 = \frac{1}{(\bar{w} - \bar{v})}$$

With, $\bar{u} = \sum_{i=1}^{n} \frac{u_i}{n}$; $\bar{v} = \sum_{i=1}^{n} \frac{v_i}{n}$; $\bar{w} = \sum_{i=1}^{n} \frac{w_i}{n}$

As MLEs hold invariance property, the MLE $\hat{R}(t)$ of $R(t)$ is obtained as

$$\hat{R}(t) = e^{-3\hat{\lambda}t} + \frac{3\hat{\lambda}\left(e^{-2\hat{\lambda}_1 t} - e^{-3\hat{\lambda}t}\right)}{3\hat{\lambda} - 2\hat{\lambda}_1} + \frac{3\hat{\lambda}\hat{\lambda}_1}{\hat{\lambda}_2 - 2\hat{\lambda}_1}\left\{\frac{\left(e^{-2\hat{\lambda}_1 t} - e^{-3\hat{\lambda}t}\right)}{(3\hat{\lambda} - 2\hat{\lambda}_1)} - \frac{\left(e^{-\hat{\lambda}_2 t} - e^{-3\hat{\lambda}t}\right)}{(3\hat{\lambda} - \hat{\lambda}_2)}\right\}.$$

5. Simulation Study

The failure times of the components of the system are generated synthetically in order to estimate the MLEs of parameters.

- **Step 1:** The failure time of the first component u_i is the minimum among the generated three exponential random variables with the parameter $\lambda = \lambda_0$.
- **Step 2:** The failure time of the second component v_i is obtained by adding u_i to the minimum of the generated two exponential random variables with the parameter $\lambda_1 = \lambda_{1_0}$.
- **Step 3:** Adding v_i to a generated exponential r. v with the parameter $\lambda_2 = \lambda_{2_0}$ in order to get failure time of the third component.
- **Step 4:** Repeat steps 1 to 3 for $n = n_0$ number of times to compute $\bar{u}, \bar{v}$ and $\bar{w}$ and using these values, $\hat{R}(t)$ are estimated at various time points t.

The Mean absolute biases (MAB) of $\hat{R}(t)$ are obtained by recurring the steps 1 to 4 for $m = 10{,}000$ simulations, using the following formula

$$MAB\left(\hat{R}(t)\right) = \frac{\sum_{k=1}^{m} |R(t) - \hat{R}_k(t)|}{m}$$

Table 2 presents the mean absolute biases of the estimated reliability function $\hat{R}(t)$ for two different sets of parameter configurations.

Table 2: Mean Absolute Biases of $\hat{R}(t)$

t	$\lambda = 0.3$, $\lambda_1 = 0.4$, $\lambda_2 = 0.5$				$\lambda = 0.6$, $\lambda_1 = 0.5$, $\lambda_2 = 0.4$			
	$n = 10$	$n = 15$	$n = 25$	$n = 45$	$n = 10$	$n = 15$	$n = 25$	$n = 45$
1	0.036242	0.028623	0.021656	0.015856	0.051126	0.040886	0.031176	0.023265
2	0.069534	0.056261	0.043454	0.032168	0.067949	0.055299	0.042961	0.032299
3	0.078791	0.064638	0.050458	0.037580	0.063348	0.051855	0.040569	0.030342
4	0.073273	0.060606	0.047561	0.035438	0.055252	0.045428	0.035559	0.026431
5	0.061825	0.051385	0.040306	0.029991	0.046876	0.038631	0.030333	0.022488
6	0.049313	0.041104	0.032079	0.023811	0.038686	0.031960	0.025191	0.018666
7	0.037940	0.031654	0.024515	0.018132	0.031192	0.025816	0.020396	0.015118
8	0.028477	0.023729	0.018203	0.013400	0.024694	0.020436	0.016155	0.011978
9	0.020998	0.017443	0.013231	0.009683	0.019299	0.015932	0.012582	0.009318
10	0.015285	0.012632	0.009462	0.006880	0.014940	0.012280	0.009669	0.007140

In above table, the results are reported for varying time points and for different values of n. The general trend observed in Table 2 is that for any fixed t the mean absolute bias of $\hat{R}(t)$ decreases as the n increases. This is expected, as larger values of n typically result in improved estimation precision due to reduced sampling variability.

6. Conclusions

This study presented a reliability analysis of a 1-out-of-3 system using the Freund bivariate exponential model, which incorporates dependency among component lifetimes resulting from progressive system stress. The expressions for the reliability function, hazard rate and mean residual life (MRL) were derived and examined under two distinct sets of failure rates: increasing and decreasing. The analysis revealed that while the reliability function consistently decreases over time, the behavior of the hazard rate and MRL depends significantly on the pattern of failure rate progression. In particular, the hazard rate increases and MRL decreases in the case of increasing failure rates, whereas the reverse trend is observed when the failure rates decrease, demonstrating the sensitivity of system behavior to the underlying stress dynamics.

Parameter estimation was carried out using maximum likelihood estimation based on data obtained from life testing of the system. A simulation study was conducted to evaluate the performance of the estimators, with results showing that the mean absolute bias decreases as sample size increases, indicating estimator consistency and precision. The findings validate the applicability of the Freund model in load-sharing environments and highlight its relevance in systems where operational stress dynamically alters component failure behavior. This framework offers practical insights for engineers and reliability analysts in designing and evaluating fault-tolerant systems across domains such as mechanical engineering, communication networks and cloud computing infrastructure.

7. References

1. Freund JE. A bivariate extension of the exponential distribution. J Am Stat Assoc. 1961;56(296):971-977.
2. Boardman TJ. On the maximum likelihood estimation of the parameters of mixed exponential distributions with applications to life testing [Dissertation]. New Brunswick (NJ): Rutgers The State University of New Jersey; 1968.
3. Block HW, Basu AP. A continuous, bivariate exponential extension. J Am Stat Assoc. 1974;69(348):1031-1037.
4. Ross SM. A model in which component failure rates depend on the working set. Nav Res Logist Q. 1984;31(2):297-300.
5. Hanagal DD. Some inference results in modified Freund's bivariate exponential distribution. Biom J. 1992;34(6):745-56.
6. Kunchur SH, Munoli SB. Estimation of reliability in Freund model for two component system. Commun Stat Theory Methods. 1994;23(11):3273-3283.
7. Kim H, Kvam PH. Reliability estimation based on system data with an unknown load share rule. Lifetime Data Anal. 2004;10(1):83-94.
8. Kvam PH, Pena EA. Estimating load-sharing properties in a dynamic reliability system. J Am Stat Assoc. 2005;100(469):262-272.
9. Park C. Parameter estimation for the reliability of load-sharing systems. IIE Trans. 2010;42(10):753-765.
10. Gurler S, Ucer BH, Bairamov I. On the mean remaining strength at the system level for some bivariate survival models based on exponential distribution. J Comput Appl Math. 2015;290:535-542.
11. Asha G, Raja AV, Ravishanker N. Reliability modelling incorporating load share and frailty. Appl Stoch Models Bus Ind. 2018;34(2):206-223.

12. Zhao X, Liu B, Liu Y. Reliability modeling and analysis of load-sharing systems with continuously degrading components. IEEE Trans Reliab. 2018;67(3):1096-1110.
13. Katagi SV, Munoli SB. Reliability assessment for 2-out-of -3 system using Freund model. Int J Sci Res Sci Technol. 2025;12(4):191-195.
14. Kundu D. A bivariate load-sharing model. J Appl Stat. 2025;52(7):1446-1469.