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Quantum codes derived from constacyclic codes over $\mathbb{Z}_3[u, v]/< u^2 = 1, v^2 = 1, uv = vu >$

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Abstract
The structural properties and construction methodologies of quantum codes (QCs) defined over the finite field $\mathbb{Z}_3$, the finite field with three elements, are examined in this work. The primary focus is on utilizing constacyclic codes within a specific algebraic structure the finite commutative non-chain ring $R = \mathbb{Z}_3 + u\mathbb{Z}_3 + v\mathbb{Z}_3 + uv\mathbb{Z}_3$, where the indeterminates u and v satisfy the relations $u^2 = 1$, $v^2 = 1$, and $uv = vu$. We provide a set of idempotent generators of the ring and define linear codes and calculated some self-inverse units. The relationship between R and $\mathbb{Z}_3^4$ is established using a gray map. Constacyclic (CC) codes are broken down into cyclic ($\mathcal{C}$) codes and negacyclic ($\mathcal{N}$) codes in order to determine the parameters of QCs over $\mathbb{Z}_3$. Several QCs of arbitrary lengths are developed as an application. In this paper, we calculated those various parameters of QCs which are better than existing parameters of QCs. This paper contains only those units which have self-inverse.

Keywords: Rate of metabolism, blood mass stream rate, warm conductivity, warm era, limited component method, Pennes Bio - Heat Model

1. Introduction
Quantum error correction (QEC), is essential to quantum computing because it reduces the flaws in quantum information caused by decoherence and different kinds of quantum disruption. The presence of quantum error correction codes (QECC) was initially established independently by Shor [19]. Calderbank *et al.* [6] released a seminal work outlining the theory for constructing QCs from classical error-correcting codes. In recent years, QECC have garnered significant attention in the literature. Some researchers have utilized the Gray images of $\mathcal{C}$ codes over finite rings.

Alahmadi *et al.* [1] and Singh et. al [18] discussed QCs from CC codes over a non-chain ring and $Z_3 + vZ_3 + \omega Z_3 + v\omega Z_3$. Bag *et al.* [5] employed QCs from skew constacyclic codes over $F_q[u, v]/< u^2 - 1, v^2 - v, uv - vu >$. Dinh *et al.* [7, 8] derived QCs from a class of CC codes over finite commutative rings and $F_p[u_1, u_2, \dots, u_s]$.

Some researchers [20, 11, 17, 16, 2, 3, 9] proposed a method for constructing QCs from C codes over $F_2 + uF_2 + u^2F_2$, $F_p[u, v, w]/< u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw >$, $F_2 + vF_2$, $F_3 + uF_3 + vF_3 + uvF_3$, $F_3 + vF_3$, and $F_p[u, v]/< u^2 - 1, v^3 - v, uv - vu >$. Ma *et al.* and Islam *et al.* [14, 12] presented constacyclic codes over $F_p + vF_p + v^2F_p$ and $F_{p^m}[v, \omega]/< v^2 - 1, \omega^2 - 1, v\omega - \omega v >$.

This work references classical QCs from CC codes recently established by various researchers over F_p [4, 13, 15, 21]. Additionally, Gowdhaman *et al.* [10] explored QCs from CC codes over the ring $F_p[u, v]/< u^3 - u, v^3 - v, uv = vu >$.In this paper, we focus on the various parameters of QCs over the field $\mathbb{Z}_3[u, v]/< u^2 = 1, v^2 = 1, uv = vu >$. In section 2.1, we present the arbitrary elements of the ring and provide several auxiliary definitions. Section 2.2 introduces the gray map defined on R. section 2.3, gives the linear codes and QCs from CC codes associated with R. Section 3 offers a collection of results that are useful in estimating the parameters of QCs accompanied by illustrative examples. Lastly, section 4 concludes our study and section 5 outlines potential avenues for future research.

2. Materials and Methods

2.1 Preliminaries

The ring $R = \mathbb{Z}_3 + u\mathbb{Z}_3 + v\mathbb{Z}_3 + uv\mathbb{Z}_3$, with $u^2 = 1$, $v^2 = 1$, and $uv = vu$, is a principal ideal ring containing 3^4 elements, with characteristic 3. The maximal ideals of R appeared in [18] which are

$\langle 2 + u + 2v \rangle, \langle 2 + u + uv \rangle, \langle u + v + 2uv \rangle, \langle 2u + 2v + 2uv \rangle$.

Notable units of R include

$1 - u + v - 2uv$, $1 + u - v - 2uv$, $1 - u - v + 2uv$,

$1 + 2u - 2v - 2uv$, $1 - 2u + 2v - 2uv$, and $1 - 2u - 2v + 2uv$.

We take ϱ as a unit of R for simplicity's sake, and observe that $\varrho^{-1} = \varrho$ in every instance. Let's define

$b_1 = 1 + u + v + uv,$

$b_2 = 1 - u + v - uv,$

$b_3 = 1 + u - v - uv,$

$b_4 = 1 - u - v + uv.$

It follows that $b_i^2 = b_i$, $b_i b_j = 0$, and $\sum_{i=1}^{4} b_i = 1$; $i, j = 1,2,3,4$ with $i \neq j$.

Utilizing the Chinese Remainder Theorem, a simple method to express the ring is as:

$R = \bigoplus_{i=1}^{4} b_i \mathbb{Z}_3$

Consequently, an arbitrary element $e = e_1 + ue_2 + ve_3 + uve_4$ of R, where $e_i \in \mathbb{Z}_3$, has a unique representation $e = \sum_{i=1}^{4} b_i k_i$, with $k_i \in Z_3$ for $1 \leq i \leq 4$.
We commence our conversation by defining a few key terms:

1. The number of places at which two vectors differ, represented by d(x,y), is the Hamming distance between $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

2. The number of nonzero x_i in a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is its Hamming weight, represented by $wt(\mathbf{x})$.

3. $\mathbf{x} \cdot \mathbf{y} = x_0 y_0 + x_1 y_1 + \cdots + x_n y_n$ is the Euclidean inner product of $\mathbf{x}$ and $\mathbf{y}$ for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$.

4. A code C is categorized as dual-containing if $C^{\perp} \subseteq C$, self-orthogonal if $C \subseteq C^{\perp}$, and self-dual if $C = C^{\perp}$.

5. A linear code C of length n over R is identified as a cyclic code if every cyclic shift of a codeword $\mathfrak{c} \in C$ remains a codeword in C. Specifically, if $c = (\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_{\mathfrak{n}-1}) \in C$, then its cyclic shift $\mathfrak{C}(c) = (\mathfrak{c}_{\mathfrak{n}-1}, \mathfrak{c}_0, \ldots, \mathfrak{c}_{\mathfrak{n}-2}) \in C$, where the operator $\mathfrak{C}$ denotes cyclic shifting.

6. The mappings from R^n to R^n, denoted by $\mathfrak{C}$, $\aleph$, and τ_{ϱ}, are defined as:

$\mathfrak{C}(\mathfrak{c}_0, \mathfrak{c}_1, \ldots \mathfrak{c}_{\mathfrak{n}-1}) = (\mathfrak{c}_{\mathfrak{n}-1}, \mathfrak{c}_0, \ldots \mathfrak{c}_{\mathfrak{n}-2}),$

$\aleph(\mathfrak{c}_0, \mathfrak{c}_1, \ldots \mathfrak{c}_{\mathfrak{n}-1}) = (-\mathfrak{c}_{\mathfrak{n}-1}, \mathfrak{c}_0, \ldots, \mathfrak{c}_{\mathfrak{n}-2}),$

$\tau_{\varrho}(\mathfrak{c}_0, \mathfrak{c}_1, \ldots \mathfrak{c}_{\mathfrak{n}-1}) = (\varrho\mathfrak{c}_{\mathfrak{n}-1}, \mathfrak{c}_0, \ldots, \mathfrak{c}_{\mathfrak{n}-2})$ respectively.

Then R is $\mathcal{C}$, $\mathcal{N}$ and ϱ-CC codes of length n if $\mathfrak{C}(C) = C$, $\aleph(C) = C$ and $\tau_{\varrho}(C) = C$ respectively.

7. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with coefficients from an arbitrary field. The polynomial is termed reciprocal or reflected (denoted p^* or p^R) if $p^*(x) = a_n + a_{n-1} x + \cdots + a_0 x^n = x^n p(x^{-1})$.

8. A polynomial $f(x)$ is defined as self-reciprocal if $p(x) = p^*(x)$.

2.2 Gray map on R

The gray map $\Psi: R \rightarrow \mathbb{Z}_3^4$ is defined as

$$\Psi(\rho) = \sum_{i=1}^{4} b_i\rho_i = (\sum_{i=1}^{4} \rho_i\, , \sum_{i=1}^{4}(-1)^{i+1}\rho_i\, , \rho_1 + \rho_2 - \rho_3 - \rho_4, \rho_1 - \rho_2 - \rho_3 + \rho_4).$$

Since Ψ is a linear map from (R^n, d_L) to $(\mathbb{Z}_3^n, d_H)$ and preserves distances as an isometric mapping.

Theorem 2.2.1 [18]: If $\mathcal{P}$ is a linear code with $|\mathcal{P}| = 3^{\mathrm{k}}$ and $\mathrm{d_L}(\mathcal{P}) = \mathrm{d}$, then $\Psi(\mathcal{P})$ is a ternary $[4\mathrm{n}, \mathrm{k}, \mathrm{d}]$ linear code, where k is the dimension of the linear code $\psi(\mathcal{P})$ and $\mathrm{d_L}$ is minimum Lee distance of the linear code $\mathcal{P}$.

Theorem 2.2.2 [18]: If $\mathcal{P}$ is a linear code, then $\Psi(\mathcal{P}^{\perp}) = (\Psi(\mathcal{P}))^{\perp}$.

2.3 Quantum codes from ϱ-CC Codes
For $i = 1,2,3,4$, let S_i be linear codes of length n over $\mathbb{Z}_3$. For a linear code $\mathcal{P}$, we define

$\mathcal{P}_1 = \{\sum_{j=1}^{4} \mathfrak{p}_j \in \mathbb{Z}_3^n | \mathfrak{p}_1 + \mathfrak{p}_2 u + \mathfrak{p}_3 v + \mathfrak{p}_4 uv \in \mathcal{P}\},$

$\mathcal{P}_2 = \{\sum_{j=1}^{4}(-1)^{j+1}\mathfrak{p}_j \in \mathbb{Z}_3^n | \mathfrak{p}_1 + \mathfrak{p}_2 u + \mathfrak{p}_3 v + \mathfrak{p}_4 uv \in \mathcal{P}\},$

$\mathcal{P}_3 = \{\mathfrak{p}_1 + \mathfrak{p}_2 - \mathfrak{p}_3 - \mathfrak{p}_4 \in \mathbb{Z}_3^n | \mathfrak{p}_1 + \mathfrak{p}_2 u + \mathfrak{p}_3 v + \mathfrak{p}_4 uv \in \mathcal{P}\},$

$\mathcal{P}_4 = \{\mathfrak{p}_1 - \mathfrak{p}_2 - \mathfrak{p}_3 + \mathfrak{p}_4 \in \mathbb{Z}_3^n | \mathfrak{p}_1 + \mathfrak{p}_2 u + \mathfrak{p}_3 v + \mathfrak{p}_4 uv \in \mathcal{P}\}.$

Clearly, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$ are linear codes over $\mathbb{Z}_3$ of length n.

Theorem 2.3.1 [18]: If $\mathcal{P}$ is a linear code, then $\Psi(\mathcal{P}) = \otimes_{\mathrm{i}=1}^{4} \mathcal{P}_{\mathrm{i}}$ and $|\mathcal{P}| = |\mathcal{P}_1||\mathcal{P}_2||\mathcal{P}_3||\mathcal{P}_4|$.

Corollary 2.3.2 [18]: If $\Psi(\mathcal{P}) = \otimes_{\mathrm{i}=1}^{4} \mathcal{P}_{\mathrm{i}}$, then $\mathcal{P} = \oplus_{\mathrm{i}=1}^{4} \mathrm{b_i}\mathcal{P}_{\mathrm{i}}$.

We explore the connections among CC codes, $\mathcal{N}$ codes, and $\mathcal{C}$ codes by analyzing various units.

Theorem 2.3.3 For $\varrho = 1 + \mathrm{u} - \mathrm{v} - 2\mathrm{uv}$, the code $\mathcal{P}$ is a ϱ-CC code if and only if $\mathcal{P}_1$, $\mathcal{P}_3$, and $\mathcal{P}_4$ are $\mathcal{N}$ codes and $\mathcal{P}_2$ is a $\mathcal{C}$ code.

Proof. For any $\xi = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in R^n$,

Where $\xi_j = \varpi_1\tau + \varpi_2\eta + \varpi_3\gamma + \varpi_4\delta$, with $\tau = (\tau_0, \tau_1, \dots, \tau_{n-1}) \in \mathcal{P}_1$, $\eta = (\eta_0, \eta_1, \dots, \eta_{n-1}) \in \mathcal{P}_2$, $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1}) \in \mathcal{P}_3$,

and $\delta = (\delta_0, \delta_1, \dots, \delta_{n-1}) \in \mathcal{P}_4$, for all $\tau_j, \eta_j, \gamma_j, \delta_j \in \mathbb{Z}_3$ for $j = 0,1, \dots, n-1$.

For the ϱ-CC code R,

$\tau_\rho(\xi) = \big((1 + u - v - 2uv)\xi_{n-1}, \xi_0, \xi_1, \dots, \xi_{n-2}\big)$

$= ((1 + u - v - 2uv)((1 + u + v + uv)\tau_{n-1} + (1 - u + v - uv)\eta_{n-1} +$

$(1 + u - v - uv)\gamma_{n-1} + (1 - u - v + uv)\delta_{n-1}), \xi_0, \xi_1, \dots, \xi_{n-2}).$

This expression simplifies to demonstrate that $\mathcal{P}_1$, $\mathcal{P}_3$, and $\mathcal{P}_4$ are $\mathcal{N}$ code while $\mathcal{P}_2$ is a $\mathcal{C}$ code of length n.
Conversely, if $\mathcal{P}_1$, $\mathcal{P}_3$, and $\mathcal{P}_4$ are $\mathcal{N}$ codes and $\mathcal{P}_2$ is a $\mathcal{C}$ code of length n,

Then $\aleph(\tau) \in \mathcal{P}_1$, $\mathrm{C}(\eta) \in \mathcal{P}_2$, $\aleph(\gamma) \in \mathcal{P}_3$, and $\aleph(\delta) \in \mathcal{P}_4$. Thus,

$(1 + u + v + uv)\aleph(\tau) + (1 - u + v - uv)\mathrm{C}(\eta) + (1 + u - v - uv)\aleph(\gamma) + (1 - u - v + uv)\aleph(\delta) \in \mathcal{P},$

$\Rightarrow \varrho(\xi) \in \mathcal{P}$. So, $\mathcal{P}$ is a ϱ-CC code.

Theorem 2.3.4 For $\varrho = 1 - \mathrm{u} + \mathrm{v} - 2\mathrm{uv}$, the code $\mathcal{P}$ is a ϱ-CC code if and only if $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ are $\mathcal{N}$ codes and $\mathcal{P}_4$ is a $\mathcal{C}$ code.

Proof. For any $\xi = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in R^n$, Where $\xi_j = \varpi_1\tau + \varpi_2\eta + \varpi_3\gamma + \varpi_4\delta$, with $\tau = (\tau_0, \tau_1, \dots, \tau_{n-1}) \in \mathcal{P}_1$, $\eta = (\eta_0, \eta_1, \dots, \eta_{n-1}) \in \mathcal{P}_2$, $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1}) \in \mathcal{P}_3$,

And $\delta = (\delta_0, \delta_1, \dots, \delta_{n-1}) \in \mathcal{P}_4$, for all $\tau_j, \eta_j, \gamma_j, \delta_j \in \mathbb{Z}_3$ for $j = 0,1, \dots, n-1$.

For the ϱ-CC code R,

$\tau_\rho(\xi) = \big((1 - u + v - 2uv)\xi_{n-1}, \xi_0, \xi_1, \dots, \xi_{n-2}\big)$

$= ((1 - u + v - 2uv)((1 + u + v + uv)\tau_{n-1} + (1 - u + v - uv)\eta_{n-1}$

$+(1 + u - v - uv)\gamma_{n-1} + (1 - u - v + uv)\delta_{n-1}), \xi_0, \xi_1, \dots, \xi_{n-2}).$

This expression simplifies to demonstrate that $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ are $\mathcal{N}$ codes while $\mathcal{P}_4$ is a $\mathcal{C}$ code of length n.
Conversely, if $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ are $\mathcal{N}$ codes and $\mathcal{P}_4$ is a $\mathcal{C}$ code of length n,

Then $\aleph(\tau) \in \mathcal{P}_1$, $\aleph(\eta) \in \mathcal{P}_2$, $\aleph(\gamma) \in \mathcal{P}_3$, and $\complement(\delta) \in \mathcal{P}_4$.

Thus,

$(1 + u + v + uv)\aleph(\tau) + (1 - u + v - uv)\aleph(\eta) + (1 + u - v - uv)\aleph(\gamma) + (1 - u - v + uv)\complement(\delta) \in \mathcal{P},$

$\Rightarrow \varrho(\xi) \in \mathcal{P}$. Thus, $\mathcal{P}$ is a ϱ-CC code.

Theorem 2.3.5 For $\varrho = 1 - u - v + 2uv$, the code $\mathcal{P}$ is a ϱ-CC code if and only if $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$ are $\mathcal{N}$ codes and $\mathcal{P}_1$ is a $\mathcal{C}$ code.

Proof. For any $\xi = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in R^n$,

Where $\xi_j = \varpi_1\tau + \varpi_2\eta + \varpi_3\gamma + \varpi_4\delta$, with $\tau = (\tau_0, \tau_1, \dots, \tau_{n-1}) \in \mathcal{P}_1$, $\eta = (\eta_0, \eta_1, \dots, \eta_{n-1}) \in \mathcal{P}_2$, $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1}) \in \mathcal{P}_3$,

And $\delta = (\delta_0, \delta_1, \dots, \delta_{n-1}) \in \mathcal{P}_4$, for all $\tau_j, \eta_j, \gamma_j, \delta_j \in \mathbb{Z}_3$ for $j = 0,1, \dots, n-1$.

For the ϱ-CC code R,

$\tau_\rho(\xi) = \big((1 - u - v + 2uv)\xi_{n-1}, \xi_0, \xi_1, \dots, \xi_{n-2}\big)$

$= ((1 - u - v + 2uv)((1 + u + v + uv)\tau_{n-1} + (1 - u + v - uv)\eta_{n-1}$

$+(1 + u - v - uv)\gamma_{n-1} + (1 - u - v + uv)\delta_{n-1}), \xi_0, \xi_1, \dots, \xi_{n-2})\dots$

This expression simplifies to demonstrate that $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$ are $\mathcal{N}$ codes while $\mathcal{P}_1$ is a $\mathcal{C}$ code of length n.
Conversely, if $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$ are $\mathcal{N}$ codes and $\mathcal{P}_1$ is a $\mathcal{C}$ code of length n,

Then $\complement(\tau) \in \mathcal{P}_1$, $\aleph(\eta) \in \mathcal{P}_2$, $\aleph(\gamma) \in \mathcal{P}_3$, and $\aleph(\delta) \in \mathcal{P}_4$.

Thus,

$(1 + u + v + uv)\complement(\tau) + (1 - u + v - uv)\aleph(\eta) + (1 + u - v - uv)\aleph(\gamma) + (1 - u - v + uv)\aleph(\delta) \in \mathcal{P},$

$\Longrightarrow \varrho(\xi) \in \mathcal{P}$. Hence, $\mathcal{P}$ is a ϱ-CC code.

Table 1

Units	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$
$1 + u - v - 2uv$	$\mathcal{N}$	$\mathcal{C}$	$\mathcal{N}$	$\mathcal{N}$
$1 - u + v - 2uv$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{C}$
$1 - u - v + 2uv$	$\mathcal{C}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
$1 + 2u - 2v - 2uv$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{C}$	$\mathcal{N}$
$1 - 2u + 2v - 2uv$	$\mathcal{N}$	$\mathcal{C}$	$\mathcal{N}$	$\mathcal{N}$
$1 - 2u - 2v + 2uv$	$\mathcal{C}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$

Theorem 2.3.6 For a ϱ-CC code, $\mathcal{P} = (b_1\hbar_1(t), b_2\hbar_2(t), b_3\hbar_3(t), b_4\hbar_4(t)) = \sum_{i=1}^{4} b_i\hbar_i(t)$, where $\hbar_i(t)$ are the generator polynomials(GP) of $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ for $i \in [1,4]_Z$ respectively. Furthermore, $|\mathcal{P}| = 3^{4n-\sum_{i=1}^{4}\deg(\hbar_i(t))}$.

Theorem 2.3.7 The dual of a ϱ-CC code has a length similar to that of the ϱ-CC code.

Theorem 2.3.8 For a ϱ-CC code $\mathcal{P}$, the dual code $\mathcal{P}^{\perp}$ is defined as follows:

1. $\mathcal{P}^{\perp} = \bigoplus_{i=1}^{4} b_i\mathcal{P}_i^{\perp}$.
2. $\mathcal{P}^{\perp} = b_1\hbar_1^*(t), b_2\hbar_2^*(t), b_3\hbar_3^*(t), b_4\hbar_4^*(t) = \sum_{i=1}^{4} b_i\hbar_i^*(t)$, where $\hbar_i^*(t)$ are the reciprocal polynomials of $t^n + 1/\hbar_i(t)$ and $t^n - 1/\hbar_i(t)$, for $i = 1,2,3$.
3. $|\mathcal{P}^{\perp}| = 3^{\sum_{i=1}^{4}\deg(h_i(t))}$.

Lemma 2.3.9 If $\mathcal{P}$ is a $\mathcal{C}$ and $\mathcal{N}$ code over $\mathbb{Z}_p$ with GP $\hbar(t)$, then $\mathcal{P}$ contains its dual code iff $t^n - r \cong 0 \bmod(\hbar(t)\hbar^*(t))$, where $r = \pm 1$.

Theorem 2.3.10 Let $\mathcal{P} = \sum_{i=1}^{4} b_i\hbar_i(t)$ be a ϱ-CC code. Then

1. If $\varrho = 1 + u - v - 2uv$, then $\mathcal{P}^{\perp} \subseteq \mathcal{P}$ iff $t^n + 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_1, \mathcal{P}_3, \mathcal{P}_4$, and $t^n - 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_2$...(*)
2. If $\varrho = 1 - u + v - 2uv$, then $\mathcal{P}^{\perp} \subseteq \mathcal{P}$ iff $t^n + 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$for $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, and $t^n - 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_1$.
3. If $\varrho = 1 - u - v + 2uv$, then $\mathcal{P}^{\perp} \subseteq \mathcal{P}$ iff $t^n + 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$, and $t^n - 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_1$.
4. If $\varrho = 1 + 2u - 2v - 2uv$, then $\mathcal{P}^{\perp} \subseteq \mathcal{P}$ iff $t^n + 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_4$, and $t^n - 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_3$.
5. If $\varrho = 1 - 2u + 2v - 2uv$, then $\mathcal{P}^{\perp} \subseteq \mathcal{P}$ iff $t^n + 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_1, \mathcal{P}_3, \mathcal{P}_4$, and $t^n - 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_2$.
6. If $\varrho = 1 + 2u - 2v + 2uv$, then $\mathcal{P}^{\perp} \subseteq \mathcal{P}$ iff $t^n + 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$, and $t^n - 1 \cong 0 \bmod(\hbar(t)\hbar^*(t))$ for $\mathcal{P}_1$.

Proof. First, we assume that (*), holds. By the above lemma, we conclude that $\mathcal{P}_i$ for $i = 1,2,3,4$ are dual-containing codes. Additionally, $b_i\mathcal{P}_i$ for $i = 1,2,3,4$ are also dual-containing codes. Thus, we have $\sum_{i=1}^{4} b_i\mathcal{P}_i^{\perp} \subseteq \sum_{i=1}^{4} b_i\mathcal{P}_i$. Consequently, $\mathcal{P}^{\perp} \subseteq \mathcal{P}$. Conversely, let $\mathcal{P}^{\perp} \subseteq \mathcal{P}$. Then, $\sum_{i=1}^{4} b_i\mathcal{P}_i^{\perp} \subseteq \sum_{i=1}^{4} b_i\mathcal{P}_i$, which implies that $b_i\mathcal{P}_i$ for $i = 1,2,3,4$ are also dual-containing codes. Therefore, $\mathcal{P}_i$ for $i = 1,2,3,4$ are dual-containing codes, and by the above lemma, (*) holds.

Similarly, the remaining parts of the theorem can be proven using analogous reasoning.

Corollary 2.3.11 Let $\mathcal{P} = \bigoplus_{i=1}^{4} b_i\mathcal{P}_i$ be a ϱ-CC code over R. Then, $\mathcal{P}^{\perp} \subseteq \mathcal{P}$ if and only if $\mathcal{P}_1^{\perp} \subseteq \mathcal{P}_1, \mathcal{P}_2^{\perp} \subseteq \mathcal{P}_2, \mathcal{P}_3^{\perp} \subseteq \mathcal{P}_3, \mathcal{P}_4^{\perp} \subseteq \mathcal{P}_4$.

Theorem 2.3.12 (CSS Construction) Let $\mathcal{P}$ be a linear code with parameters $[n, k, d]$ over $\mathbb{Z}_3$. A QCs with parameters $[n, 2k - n, d_L]$ can be acquired if $\mathcal{P}^{\perp} \subseteq \mathcal{P}$.

3 Results and Discussion

Utilizing Theorems 2.3.9 and 2.3.10, the following QECC construction can be derived.

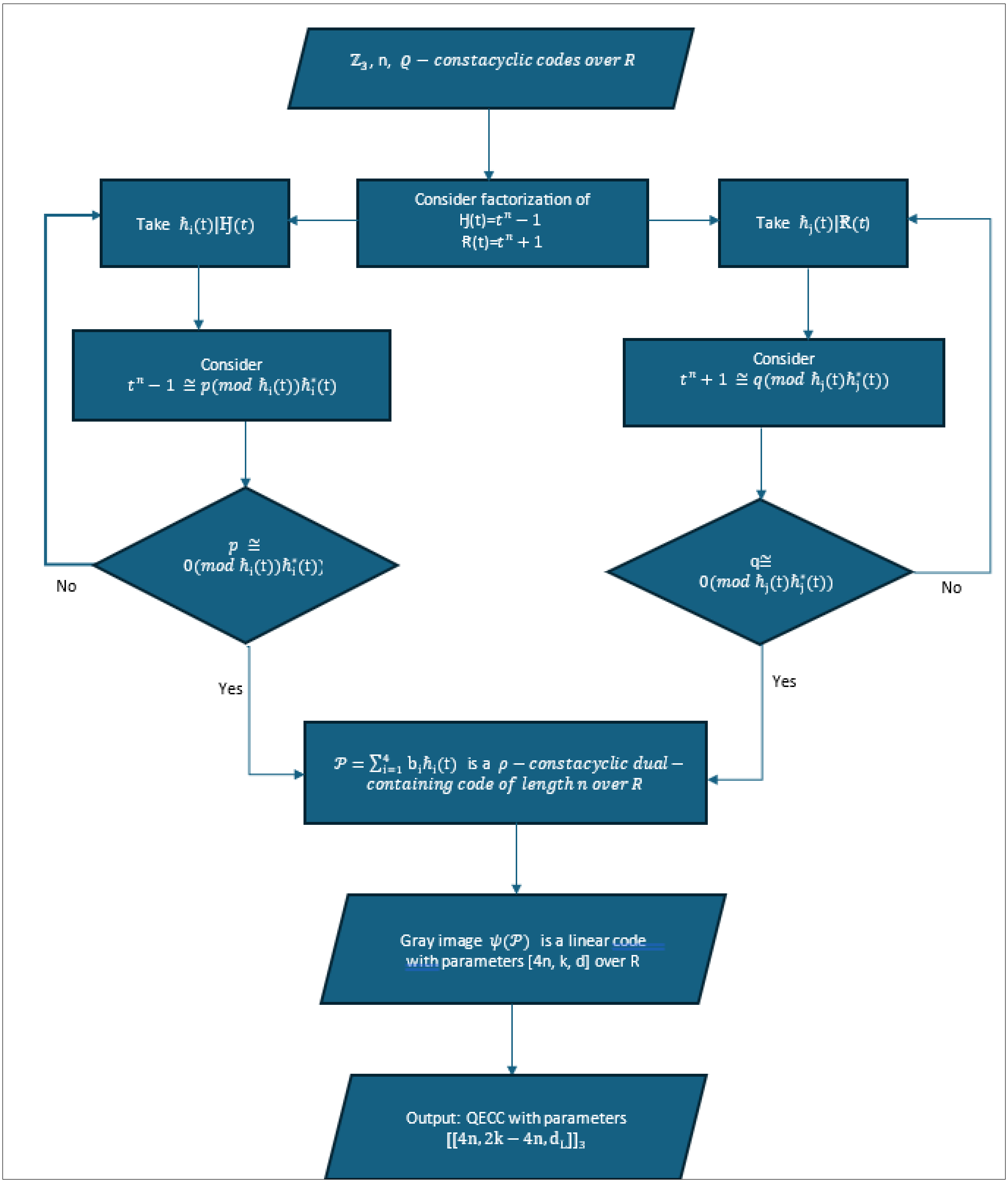
Theorem 3.1.1 If $\mathcal{P} = \bigoplus_{i=1}^{4} b_i\mathcal{P}_i = \sum_{i=1}^{4} b_i\hbar_i(t)$ is a ϱ-CC code over R and if $\mathcal{P}^{\perp} \subseteq \mathcal{P}$, then ∃ a QCs with parameters $[4n, 2k - 4n, d_L]$.

Remark 3.1.2 The construction of QECC from ϱ-CC codes presents a more advantageous approach compared to the construction of QECC from $\mathcal{C}$ codes. When utilizing $\mathcal{C}$ codes for QECC, our sole option is to set ϱ to 1. In contrast, constructing QECC from ϱ-CC codes offers a broader choices of possible values for ϱ. For instance, when considering the ring R, we can select ϱ from set

$\{1 - u + v - 2uv, 1 + u - v - 2uv, 1 - u - v + 2uv, 1 + 2u - 2v - 2uv,$

$1 - 2u + 2v - 2uv, 1 - 2u - 2v + 2uv\}$, rather than being restricted to the single value of 1.

We introduce a flowchart for building QECCs using ϱ-CC codes over R.



3.2 Examples

Example 3.2.1 Given that $R = \mathbb{Z}_3 + u\mathbb{Z}_3 + v\mathbb{Z}_3 + uv\mathbb{Z}_3$, in which $u^2 = 1, v^2 = 1, uv = vu$, and n=15. It follows that $t^{15} - 1 = (t+2)^3(t^4 + t^3 + t^2 + t + 1)^3$

and $t^{15} + 1 = (t+1)^3(t^4 + 2t^3 + t^2 + 2t + 1)^3$.

Consider a (1+u-v-2uv)-CC code of length 15 over R. The GP of $\mathcal{P}$ is

$\hbar(t) = \sum_{i=1, i\neq 2}^{4} b_i (t+1)^2 + b_2 (t+2)^2$.

Let $\hbar_1(t) = \hbar_3(t) = \hbar_4(t) = (t+1)^2$ and $\hbar_2(t) = (t+2)^2$. Moreover, $\Psi(\mathcal{P})$ has parameter [60,52,3]. Then, using theorem 2.3.13, we derive the QCs with parameters [60,44,$\geq$ 3].

Example 3.2.2 Let $R = \mathbb{Z}_3 + u\mathbb{Z}_3 + v\mathbb{Z}_3 + uv\mathbb{Z}_3$ with $u^2 = 1, v^2 = 1, uv = vu$ and n=3. Then, $t^3 - 1 = (t+2)^3$ and $t^3 + 1 = (t+1)(t^2 - t + 1)$. Let $\mathcal{P}$ be a (1-2u-2v+2uv)-CC code of length 3 over R.

Take $\hbar_1(t) = (t+2)$ and $\hbar_2(t) = \hbar_3(t) = \hbar_4(t) = (t+1)$ then

$\hbar(t) = \sum_{i=1, i\neq 1}^{4} b_i(t+1) + b_1((t+2)$ be the GP of $\mathcal{P}$. Then $\Psi(\mathcal{P})$ is a linear code with parameter [12, 8, 2]. Then by theorem 2.3.13, we get the QCs with parameters [12, 4, ≥ 2].

4 Conclusion

In our study, we identified the units that are self-inverse and utilized them to analyze the QCs of the ring $R = \mathbb{Z}_3 + u\mathbb{Z}_3 + v\mathbb{Z}_3 + uv\mathbb{Z}_3$, with $u^2 = 1$, $v^2 = 1$, and $uv = vu$.

5 Open Problem

Future investigations could delve into additional QCs over Z_3 by employing different gray maps on $\mathbb{Z}_3 + u\mathbb{Z}_3 + v\mathbb{Z}_3 + uv\mathbb{Z}_3$ while adhering to the same constraints. Moreover, it would be beneficial to explore QCs in this ring under varying conditions and to compute QCs using alternative rings.

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