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On fixed point theorems within multiplicative $\mathcal{S}$-metric space

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Abstract
In this paper, we establish some fixed point results by employing an altering distance function for mapping that fulfill certain novel contractive conditions in a complete multiplicative $\mathcal{S}$-metric space.

Keywords: Fixed point, altering distance function, contractive conditions, multiplicative $\mathcal{S}$-metric space

Introduction

Banach [1] contraction principle has been a very advantageous and efficacious means in nonlinear analysis. Various authors have generalized Banach contraction principle in different spaces. Singxi *et al.* [18] and Sastry *et al.* [10] studied some common fixed point theorems for different mappings on a 2-metric space. Dhage [4] proved fixed point results in D-metric space. A. E. Bashirov *et al.* [2] introduce a new kind of space called multiplicative metric space in the year 2008 and studied some properties of multiplicative derivatives and multiplicative intergrals.

Definition 1.1 [2]: "Let $\mathcal{X}$ be a non-empty set. A multiplicative metric is a mapping on $d: X \times X \rightarrow \mathbb{R}^+$ satisfying the following axioms:

1. $d(u,v) \geq 1$ for all $u, v \in \mathcal{X}$ and $d(u,v) = 1$ if and only if $u = v$,
2. $d(u,v) = d(v,u)$ for all $u, v \in \mathcal{X}$,
3. $d(u,v) \leq d(u,w).d(w,v)$, for all $u, v \in \mathcal{X}$.

Then the mapping d together with $\mathcal{X}$ that is, $(\mathcal{X}, d)$ is a multiplicative metric space." In 2012, Ozavsar and Cevikel [8] introduced the concept of multiplicative contraction mappings and proved some fixed point theorems of such mappings on a complete multiplicative metric space.

Definition 1.2 [8]: "Let $(\mathcal{X}, d)$ be a multiplicative metric space. A mapping $f: X \rightarrow X$ is called a multiplicative contraction if there exist a real constant $\lambda \in [0,1)$ such that

$$d(fu, fv) \leq d(u,v)^{\lambda}, \text{ for all } u, v \in \mathcal{X}."$$

Theorem 1.3 [8]: "Let $(\mathcal{X}, d)$ be a multiplicative metric space and let $f: X \rightarrow X$ be a multiplicative contraction. If $(\mathcal{X}, d)$ is complete then, f has a unique fixed point."
In 2012, Sedghi *et. al.* [11] establish the concept of $\mathcal{S}$-metric space as a generalization of G-metric space and metric space.

Definition 1.4 [11]: "Let $\mathcal{X}$ be a non-empty set. An $\mathcal{S}$-metric on X is a function $\mathcal{S}$: $\mathcal{X} \times \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ satisfying the following axioms:

1. $\mathcal{S}(u,v,w)=0$ if and only if $u=v=w$,
2. $\mathcal{S}(u,v,w)\le \mathcal{S}(u,u,a)+\mathcal{S}(v,v,a)+\mathcal{S}(w,w,a)$, for all $u,v,w\in\mathcal{X}$

An $\mathcal{S}$-metric space is a pair $(\mathcal{X},\mathcal{S})$ where $\mathcal{S}$ is a metric on $\mathcal{X}$."
In the setting of $\mathcal{S}$-metric space they proved several fixed and common fixed point theorems (see [5], [12]-[16]).
In 2021, Naga Raju [9] introduce the concept of multiplicative $\mathcal{S}$-metric space and studied its topological properties.

Definition 1.5 [9]: "Let $\mathcal{X}$ be a non-empty set. We say that the function $\mathcal{S}:\mathcal{X}\times\mathcal{X}\times\mathcal{X}\to[0,\infty)$ is a multiplicative $\mathcal{S}$-metric on $\mathcal{X}$ iff it satisfies the following axioms:

1. $\mathcal{S}(u,v,w)\ge 1$,
2. $\mathcal{S}(u,v)=1$ if and only if $u=v=w$,
3. $\mathcal{S}(u,v,w)\le \mathcal{S}(u,u,a).\mathcal{S}(v,v,a).\mathcal{S}(w,w,a)$, for all $u,v,w,a\in\mathcal{X}$.

Then the mapping $(\mathcal{X},\mathcal{S})$ is called a multiplicative $\mathcal{S}$-metric space."

Definition 1.6 [9]: "We say that a sequence $\{u_n\}$ in a multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$ multiplicative $\mathcal{S}$-convergent to some $\alpha\in\mathcal{X}$ iff for each $\in_0>1$, there exist $H\in\mathbb{N}$ such that $\mathcal{S}(u_n,u_n,u)<\in_0$, for all $n\ge H$."

Definition 1.7 [9]: "We say that a sequence $\{u_n\}$ in a multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$ multiplicative $\mathcal{S}$-Cauchy sequence in $\mathcal{X}$ iff for each $\in_0>1$, there exist $H\in\mathbb{N}$ such that $\mathcal{S}(u_n,u_n,u_m)<\in_0$, for all $n,m\ge H$."

Definition 1.8 [9]: "We say that a multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$ is multiplicative $\mathcal{S}$-complete iff every multiplicative $\mathcal{S}$-Cauchy sequence in $\mathcal{X}$ is multiplicative $\mathcal{S}$-convergent in $\mathcal{X}$."

Definition 1.9 [9]: "Let $(\mathcal{X},\mathcal{S})$ and $(\mho,\mathcal{S}')$ be two multiplicative $\mathcal{S}$-metric spaces. Then we say that $f:\mathcal{X}\to\mho$ is multiplicative $\mathcal{S}$-continuous at some point $\theta\in\mathcal{X}$ iff for every $r>1$, there exists $\eta>1$such that $f(B(\theta,\eta))\subset B(f(\theta),r)$. Thus, we say that f is multiplicative $\mathcal{S}$-continuous at every point of $\mathcal{X}$."

Lemma 1.10 [9]: "In multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$ we have $\mathcal{S}(u,,u,v)=\mathcal{S}(v,v,u)$ for all $u,v\in\mathcal{X}$."

Lemma 1.11 [9] "In multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$, $u_n\to u$ iff $\mathcal{S}(u_n,u_n,u)\to 1$, as $n\to\infty$."

Theorem 1.12 [9] "In multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$, if there exist two sequences $\{u_n\}$ and $\{v_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty}u_n=u$ and $\lim_{n\to\infty}v_n=v$ then $\lim_{n\to\infty}\mathcal{S}(u_n,u_n,v_n)=\mathcal{S}(u,u,v)$."

Theorem 1.13 [9]: "In multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$, $\{u_n\}$ is a multiplicative $\mathcal{S}$-Cauchy sequence in $\mathcal{X}$ iff $\mathcal{S}(u_n,u_n,u_m)\to 1$ as $m\to\infty$."

Definition 1.14 [6]: "Let f and g be two mappings of a metric space $(\mathcal{X},d)$ into itself. Then f and g are said to be weakly compatible if they commutes at coincident points, that is, if $ft=gt$ for some $t\in\mathcal{X}$ implies that $fgt=gft$."

Definition 1.15 [7]: "A function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is called an altering distance function if the following property is satisfied:

- (Θ_1) $\phi(t)=1$ if and only if $t=1$,
- (Θ_2) ϕ is monotonically non-decreasing function,
- (Θ_3) ϕ is a continuous function."

In our result we use the following class of function.

$\Phi=\{\phi:[1,\infty)\to[1,\infty):\phi$ is an altering distance function$\}$

$\Psi=\{\psi:[1,\infty)\to[1,\infty)$: for any sequence $\{u_n\}$ in $[1,\infty)$ with $u_n\to t>1$,

$\lim_{n\to\infty}\psi(u_n)>1\}$.

Note that Ψ is non empty, since $\psi(t)=e^t$ for $t\in[1,\infty)$. Thus $\psi\in\Psi$."

Remark: Clearly for $\psi\in\Psi$, $\psi(t)>1$ for $t>1$ and $\psi(1)$ need not be equal to 1.

2. Main Results

Theorem 2.1: Let $\mathcal{T}$ be a multiplicative $\mathcal{S}$-continuous mapping of a complete multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$ into itself such that for all $u,v \in \mathcal{X}$

$$\phi(\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v)) \leq \frac{\phi(M(u,u,v))}{\psi(N(u,u,v))}, \tag{2.1}$$

where $\phi \in \Phi$ and $\psi \in \Psi$,

$$M(u,u,v) = max\left\{\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)}, \frac{\mathcal{S}(v,v,\mathcal{T}u)\mathcal{S}(u,u,\mathcal{T}v)}{\mathcal{S}(u,u,v)\mathcal{S}(v,v,\mathcal{T}v)}, \mathcal{S}(u,u,v)\right\} \text{ and}$$

$$N(u,u,v) = max\left\{\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)}, \mathcal{S}(u,u,v)\right\}.$$

Then $\mathcal{T}$ has a unique fixed point.

Proof: Let u_0 be an arbitrary point. Then there exists $u_1 \in \mathcal{X}$ such that $u_1 = \mathcal{T}u_0$. So we can define a sequence $\{u_n\}$ in $\mathcal{X}$ such that $u_{n+1} = \mathcal{T}u_n$ for $n \geq 0$.
If there exists some $n \in \mathbb{N}$ such that, $u_{n+1} = u_n$. Then we have $u_{n+1} = \mathcal{T}u_n = u_n$, which implies that u_n is a fixed point of $\mathcal{T}$.
Suppose that $u_{n+1} \neq u_n$, that is $\mathcal{S}(u_{n+1},u_{n+1},u_n) \neq 1$ for all n. Then from (2.1), we have

$$\phi\big(\mathcal{S}(u_n,u_n,u_{n+1})\big) = \phi\big(\mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n)\big)$$

$$\leq \frac{\phi(M(u_{n-1},u_{n-1},u_n))}{\psi(N(u_{n-1},u_{n-1},u_n))},$$

Where

$$M(u_{n-1},u_{n-1},u_n)$$

$$= max\left\{\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_n,u_n,\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}, \frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_n,u_n,\mathcal{T}u_n)}, \mathcal{S}(u_{n-1},u_{n-1},u_n)\right\}$$

$$= max\left\{\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_n,u_n,u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}, \frac{\mathcal{S}(u_n,u_n,u_n)\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_n,u_n,u_{n+1})}, \mathcal{S}(u_{n-1},u_{n-1},u_n)\right\}$$

$$= max\{\mathcal{S}(u_n,u_n,u_{n+1}), \mathcal{S}(u_{n-1},u_{n-1},u_n), \mathcal{S}(u_{n-1},u_{n-1},u_n)\} = max\{\mathcal{S}(u_n,u_n,u_{n+1}), \mathcal{S}(u_{n-1},u_{n-1},u_n)\}$$

and

$$N(u_{n-1},u_{n-1},u_n) = max\left\{\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_n,u_n,\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}, \mathcal{S}(u_{n-1},u_{n-1},u_n)\right\}$$

$$= max\left\{\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_n,u_n,u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}, \mathcal{S}(u_{n-1},u_{n-1},u_n)\right\}$$

$$= max\{\mathcal{S}(u_n,u_n,u_{n+1}), \mathcal{S}(u_{n-1},u_{n-1},u_n)\}$$

Therefore, we have

$$\phi(\mathcal{S}(u_n,u_n,u_{n+1})) \leq \frac{\phi(max\{\mathcal{S}(u_n,u_n,u_{n+1}),\mathcal{S}(u_{n-1},u_{n-1},u_n)\})}{\psi(max\{\mathcal{S}(u_n,u_n,u_{n+1}),\mathcal{S}(u_{n-1},u_{n-1},u_n)\})}. \tag{2.2}$$

If $\mathcal{S}(u_n,u_n,u_{n+1}) > \mathcal{S}(u_{n-1},u_{n-1},u_n)$, then from (2.2), we have

$$\phi(\mathcal{S}(u_n,u_n,u_{n+1})) \leq \frac{\phi(\mathcal{S}(u_n,u_n,u_{n+1}))}{\psi(\mathcal{S}(u_n,u_n,u_{n+1}))},$$

that is, $\psi(\mathcal{S}(u_n,u_n,u_{n+1})) \leq 1$, which is a contradiction. So, we have $\mathcal{S}(u_n,u_n,u_{n+1}) \leq \mathcal{S}(u_{n-1},u_{n-1},u_n)$, which implies $\{\mathcal{S}(u_n,u_n,u_{n+1})\}$ is a decreasing sequence. Then the inequality (2.2) yields that

$$\phi(\mathcal{S}(u_n,u_n,u_{n+1})) \leq \frac{\phi(\mathcal{S}(u_{n-1},u_{n-1},u_n))}{\psi(\mathcal{S}(u_{n-1},u_{n-1},u_n))}. \tag{2.3}$$

Since $\{\mathcal{S}(u_n,u_n,u_{n+1})\}$ is a decreasing sequence of real numbers and it is bounded below, there exists $r \geq 1$ such that

$$\mathcal{S}(u_n,u_n,u_{n+1}) \to r \text{ as } n \to \infty \tag{2.4}$$

Now we shall show that $r = 1$. Assume that $r > 1$. Taking limit on both sides of (2.3) and using (2.4), the property of ψ and continuity of ϕ, we get

$$\phi(r) \leq \frac{\phi(r)}{\lim_{n\to\infty} \psi(\mathcal{S}(u_{n-1}, u_{n-1}, u_n))},$$

which implies that $\lim_{n\to\infty} \psi(\mathcal{S}(u_{n-1}, u_{n-1}, u_n)) \leq 1$, which by the property of ψ, is a contradiction. Therefore,

$$\mathcal{S}(u_n, u_n, u_{n+1}) \to 1 \text{ as } n \to \infty \tag{2.5}$$

Next we claim that $\{u_n\}$ is a multiplicative $\mathcal{S}$-Cauchy sequence. Suppose that $\{u_n\}$ is not a multiplicative $\mathcal{S}$-Cauchy sequence. Then there exists an $\in_0 > 1$ for which we can find two subsequences of positive integers $\{m(c)\}$ and $\{n(c)\}$ such that for all positive integers c, $n(c) > m(c) \geq c$ and

$$\mathcal{S}(u_{m(c)}, u_{m(c)}, u_{n(c)}) \geq \in_0.$$

Assume that $n(c)$ is the smallest positive integer, we get $n(c) > m(c) \geq c$,

$$\mathcal{S}(u_{n(c)}, u_{n(c)}, u_{m(c)}) \geq \in_0 \text{ and } \mathcal{S}\big(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)}\big) < \in_0. \tag{2.6}$$

Now,

$$\in_0 \leq \mathcal{S}\big(u_{m(c)}, u_{m(c)}, u_{n(c)}\big) = \mathcal{S}\big(u_{n(c)}, u_{n(c)}, u_{m(c)}\big)$$

$$\leq \mathcal{S}\big(u_{n(c)}, u_{n(c)}, u_{n(c)-1}\big)\mathcal{S}\big(u_{n(c)}, u_{n(c)}, u_{n(c)-1}\big)\mathcal{S}\big(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)}\big)$$

Taking limit as $c \to \infty$ and using (2.5), we get

$$\lim_{c\to\infty} \mathcal{S}\big(u_{m(c)}, u_{m(c)}, u_{n(c)}\big) = \in_0. \tag{2.7}$$

Again,

$$\mathcal{S}\big(u_{m(c)-1}, u_{m(c)-1}, u_{n(c)-1}\big)$$

$$\leq \mathcal{S}(u_{m(c)-1}, u_{m(c)-1}, u_{m(c)}).\mathcal{S}(u_{m(c)-1}, u_{m(c)-1}, u_{m(c)}).\ \mathcal{S}\big(u_{m(c)}, u_{m(c)}, u_{n(c)-1}\big).$$

Taking limit as $c \to \infty$ ad using (2.5) and (2.6), we get

$$\lim_{c\to\infty} \mathcal{S}\big(u_{m(c)-1}, u_{m(c)-1}, u_{n(c)-1}\big) = \lim_{c\to\infty} \mathcal{S}\big(u_{m(c)}, u_{m(c)}, u_{n(c)-1}\big)$$

$$= \lim_{c\to\infty} \mathcal{S}\big(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)}\big).$$

Now,

$$\mathcal{S}\big(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)}\big) \leq \mathcal{S}\big(u_{n(c)-1}, u_{n(c)-1}, u_{n(c)}\big).\mathcal{S}\big(u_{n(c)-1}, u_{n(c)-1}, u_{n(c)}\big)$$

$$.\mathcal{S}\big(u_{n(c)}, u_{n(c)}, u_{m(c)}\big),$$

Taking limit as $c \to \infty$ in above inequalities and using (2.5)-(2.7), we get

$$\lim_{c\to\infty} \mathcal{S}\big(u_{m(c)-1}, u_{m(c)-1}, u_{n(c)-1}\big) = \in_0. \tag{2.8}$$

Again,

$$\mathcal{S}\big(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)}\big)$$

$$\leq \mathcal{S}(u_{n(c)-1}, u_{n(c)-1}, u_{n(c)}).\mathcal{S}(u_{n(c)-1}, u_{n(c)-1}, u_{n(c)}).\ \mathcal{S}\big(u_{n(c)}, u_{n(c)}, u_{m(c)}\big).$$

And

$$\mathcal{S}\big(u_{m(c)-1}, u_{m(c)-1}, u_{n(c)-1}\big)$$

$\leq \mathcal{S}(u_{m(c)-1}, u_{m(c)-1}, u_{m(c)}).\mathcal{S}(u_{m(c)-1}, u_{m(c)-1}, u_{m(c)}).\mathcal{S}(u_{m(c)}, u_{m(c)}, u_{n(c)-1})$.

Taking limit as $c \to \infty$ in above inequalities and using (2.5) and (2.8), we get

$$\lim_{c\to\infty} \mathcal{S}(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)}) = \epsilon_0. \tag{2.9}$$

Similarly, we have

$$\lim_{c\to\infty} \mathcal{S}(u_{m(c)-1}, u_{m(c)-1}, u_{n(c)}) = \epsilon_0. \tag{2.10}$$

Let

$$M(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)-1}) =$$

$$max\left\{\begin{matrix}\frac{\mathcal{S}(u_{n(c)-1},u_{n(c)-1},\mathcal{T}u_{n(c)-1})\mathcal{S}(u_{m(c)-1},u_{m(c)-1},\mathcal{T}u_{m(c)-1})}{\mathcal{S}(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1})}, \frac{\mathcal{S}(u_{n(c)-1},u_{n(c)-1},\mathcal{T}u_{m(c)-1})\mathcal{S}(u_{m(c)-1},u_{m(c)-1},\mathcal{T}u_{n(c)-1})}{\mathcal{S}(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1})\mathcal{S}(u_{m(c)-1},u_{m(c)-1},\mathcal{T}u_{m(c)-1})},\\ \mathcal{S}(u_{m(c)-1}, u_{m(c)-1}, u_{n(c)-1})\end{matrix}\right\}$$

which implies

$$max\left\{\begin{matrix}\frac{\mathcal{S}(u_{n(c)-1},u_{n(c)-1},u_{n(c)})\mathcal{S}(u_{m(c)-1},u_{m(c)-1},u_{m(c)})}{\mathcal{S}(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1})}, \frac{\mathcal{S}(u_{n(c)-1},u_{n(c)-1},u_{m(c)})\mathcal{S}(u_{m(c)-1},u_{m(c)-1},u_{n(c)})}{\mathcal{S}(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1})\mathcal{S}(u_{m(c)-1},u_{m(c)-1},u_{m(c)})},\\ \mathcal{S}(u_{m(c)-1}, u_{m(c)-1}, u_{n(c)-1})\end{matrix}\right\} \tag{2.11}$$

And

$$N(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)-1})$$

$$= max\left\{\frac{\mathcal{S}(u_{n(c)-1},u_{n(c)-1},\mathcal{T}u_{n(c)-1})\mathcal{S}(u_{m(c)-1},u_{m(c)-1},\mathcal{T}u_{m(c)-1})}{\mathcal{S}(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1})}, \mathcal{S}(u_{m(c)-1}, u_{m(c)-1}, u_{n(c)-1})\right\}$$

$$= max\left\{\frac{\mathcal{S}(u_{n(c)-1},u_{n(c)-1},u_{n(c)})\mathcal{S}(u_{m(c)-1},u_{m(c)-1},u_{m(c)})}{\mathcal{S}(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1})}, \mathcal{S}(u_{m(c)-1}, u_{m(c)-1}, u_{n(c)-1})\right\}. \tag{2.12}$$

Letting $c \to \infty$ in (2.11) and (2.12), using equations (2.5)-(2.10), we have

$$\lim_{c\to\infty} M(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)-1}) = max\{1, \epsilon_0, \epsilon_0\} = \epsilon_0 \tag{2.13}$$

and

$$\lim_{c\to\infty} N(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)-1}) = max\{1, \epsilon_0, \epsilon_0\} = \epsilon_0 \tag{2.14}$$

From (2.1), using (2.11) and (2.12), we have

$$\phi\left(\mathcal{S}(u_{m(c)}, u_{m(c)}, u_{n(c)})\right) = \phi\left(\mathcal{S}(\mathcal{T}u_{m(c)-1}, \mathcal{T}u_{m(c)-1}, \mathcal{T}u_{n(c)-1})\right)$$

$$\leq \frac{\phi\left(M(u_{n(c)-1},u_{n(c)-1},u_{m(c)-1})\right)}{\psi\left(N(u_{n(c)-1},u_{n(c)-1},u_{m(c)-1})\right)}.$$

Taking limit on both sides and using (2.6), (2.13) and (2.14), the property of ψ and continuity of ϕ, we have

$$\phi(\epsilon_0) \leq \frac{\phi(\epsilon_0)}{\lim_{c\to\infty} \psi\left(N(u_{n(c)-1},u_{n(c)-1},u_{m(c)-1})\right)},$$

that is, $\lim_{c\to\infty} \psi\left(N(u_{n(c)-1}, u_{n(c)-1}, u_{m(c)-1})\right) \leq 1$, which is contradiction by the property of ψ. Thus, $\{u_n\}$ is a multiplicative $\mathcal{S}$-Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is multiplicative $\mathcal{S}$-complete, there exists $w \in \mathcal{X}$ such that $\lim_{n\to\infty} u_n = w$. Then using multiplicative $\mathcal{S}$-continuity of $\mathcal{T}$, we get

$$\mathcal{T}w = \mathcal{T}\left(\lim_{n\to\infty} u_n\right) = \lim_{n\to\infty} u_{n+1} = w.$$

Hence w is a fixed point of $\mathcal{T}$.
Finally, we shall prove the uniqueness of the fixed point of $\mathcal{T}$. Suppose that w and r $(w \neq r)$ be any fixed point of $\mathcal{T}$. Consider

$$\phi(\mathcal{S}(w,w,r)) = \phi(\mathcal{S}(\mathcal{T}w,\mathcal{T}w,\mathcal{T}r)) \leq \frac{\phi(M(w,w,r))}{\psi(N(w,w,r))},$$

where $\phi \in \Phi$ and $\psi \in \Psi$,

$$M(w,w,r) = max\left\{\frac{\mathcal{S}(w,w,\mathcal{T}w)\mathcal{S}(r,r,\mathcal{T}r)}{\mathcal{S}(w,w,r)}, \frac{\mathcal{S}(r,r,\mathcal{T}w)\mathcal{S}(w,w,\mathcal{T}r)}{\mathcal{S}(w,w,r)\mathcal{S}(r,r,\mathcal{T}r)}, \mathcal{S}(w,w,r)\right\}$$

$$= max\left\{\frac{\mathcal{S}(w,w,w)\mathcal{S}(r,r,r)}{\mathcal{S}(w,w,r)}, \frac{\mathcal{S}(r,r,w)\mathcal{S}(w,w,r)}{\mathcal{S}(w,w,r)\mathcal{S}(r,r,r)}, \mathcal{S}(w,w,r)\right\}$$

$$= \mathcal{S}(w,w,r)$$

and

$$N(u,u,v) = max\left\{\frac{\mathcal{S}(w,w,\mathcal{T}w)\mathcal{S}(r,r,\mathcal{T}r)}{\mathcal{S}(w,w,r)}, \mathcal{S}(w,w,r)\right\} = \mathcal{S}(w,w,r).$$

Therefore, we have

$$\phi(\mathcal{S}(w,w,r)) = \phi(\mathcal{S}(\mathcal{T}w,\mathcal{T}w,\mathcal{T}r)) \leq \frac{\phi(\mathcal{S}(w,w,r))}{\psi(\mathcal{S}(w,w,r))},$$

which implies that $\psi\big(\mathcal{S}(w,w,r)\big) \leq 1$, which is contraction by definition of ψ. Hence $w = r$.

Therefore $\mathcal{T}$ has a unique fixed point.
This completes the proof.
Next we prove the following result without the condition of multiplicative $\mathcal{S}$-continuity of $\mathcal{T}$.

Theorem 2.2: Let $\mathcal{T}$ be a mapping of a complete multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$ into itself such that for all $u,v \in \mathcal{X}$

$$\phi(\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v)) \leq \frac{\phi(M(u,u,v))}{\psi(N(u,u,v))},$$

where $\phi \in \Phi$ and $\psi \in \Psi$,

$$M(u,u,v) = max\left\{\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)}, \frac{\mathcal{S}(v,v,\mathcal{T}u)\mathcal{S}(u,u,\mathcal{T}v)}{\mathcal{S}(u,u,v)\mathcal{S}(v,v,\mathcal{T}v)}, \mathcal{S}(u,u,v)\right\} \text{ and}$$

$$N(u,u,v) = max\left\{\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)}, \mathcal{S}(u,u,v)\right\}.$$

Then $\mathcal{T}$ has a unique fixed point.

Proof: From the proof of Theorem 2.1 $\{u_n\}$ is a multiplicative $\mathcal{S}$-Cauchy sequence in $\mathcal{X}$, hence there exists $w \in \mathcal{X}$ such that

$$\lim_{n\to\infty} u_n = w.$$

Suppose that $\mathcal{T}w = w$, that is, $\mathcal{S}(w,w,\mathcal{T}w) > 1$. Consider

$$\phi(\mathcal{S}(\mathcal{T}u_n,\mathcal{T}u_n,\mathcal{T}w)) \leq \frac{\phi(M(u_n,u_n,w))}{\psi(N(u_n,u_n,w))} \quad (2.15)$$

Where

$$M(u_n,u_n,w) = max\left\{\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(w,w,\mathcal{T}w)}{\mathcal{S}(u_n,u_n,w)}, \frac{\mathcal{S}(w,w,\mathcal{T}u_n)\mathcal{S}(u_n,u_n,\mathcal{T}w)}{\mathcal{S}(u_n,u_n,w)\mathcal{S}(w,w,\mathcal{T}w)}, \mathcal{S}(u_n,u_n,w)\right\} \quad (2.16)$$

$$= max\left\{\frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(w,w,\mathcal{T}w)}{\mathcal{S}(u_n,u_n,w)}, \frac{\mathcal{S}(w,w,u_{n+1})\mathcal{S}(u_n,u_n,\mathcal{T}w)}{\mathcal{S}(u_n,u_n,w)\mathcal{S}(w,w,\mathcal{T}w)}, \mathcal{S}(u_n,u_n,w)\right\}$$

And

$$N(u_n,u_n,w) = max\left\{\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(w,w,\mathcal{T}w)}{\mathcal{S}(u_n,u_n,w)}, \mathcal{S}(u_n,u_n,w)\right\}$$

$= max\left\{\frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(w,w,\mathcal{T}w)}{\mathcal{S}(u_n,u_n,w)}, \mathcal{S}(u_n,u_n,w)\right\}$. (2.17)

Letting $n \to \infty$ in (2.16) and (2.17), we have

$$\lim_{c\to\infty} M(u_n,u_n,\mathcal{T}w) = \mathcal{S}(w,w,\mathcal{T}w) > 1$$

And

$$\lim_{c\to\infty} N(u_n,u_n,\mathcal{T}w) = \mathcal{S}(w,w,\mathcal{T}w) > 1.$$

Again letting $n \to \infty$ in (2.15) using (2.16), (2.17) and property of ϕ and ψ, we have

$$\phi(\mathcal{S}(w,w,\mathcal{T}w)) \leq \frac{\phi(\mathcal{S}(w,w,\mathcal{T}w))}{\lim_{n\to\infty}\psi(N(u_n,u_n,w))},$$

Which implies that $\lim_{n\to\infty}\psi(N(u_n,u_n,w)) < 1$, which is contradiction by property of ψ.
Therefore $\mathcal{T}w = w$ and hence w is a fixed point of $\mathcal{T}$.
Uniqueness easily follows from Theorem 2.1. This completes the proof.

Corollary 2.3: Let $\mathcal{T}$ be a mapping of a complete multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$ into itself such that for all $u,v \in \mathcal{X}$

$$\phi(\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v)) \leq \frac{\phi(N(u,u,v))}{\psi(N(u,u,v))},$$

Where $\phi \in \Phi$ and $\psi \in \Psi$, and

$$N(u,u,v) = max\left\{\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)}, \mathcal{S}(u,u,v)\right\}.$$

Then $\mathcal{T}$ has a unique fixed point.

Corollary 2.4: Let $\mathcal{T}$ be a mapping of a complete multiplicative $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$ into itself such that for all $u,v \in \mathcal{X}$ and for some $k \in (0,1)$

$$\phi(\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v)) \leq k\, max\left\{\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)}, \mathcal{S}(u,u,v)\right\}$$

Then $\mathcal{T}$ has a unique fixed point.

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