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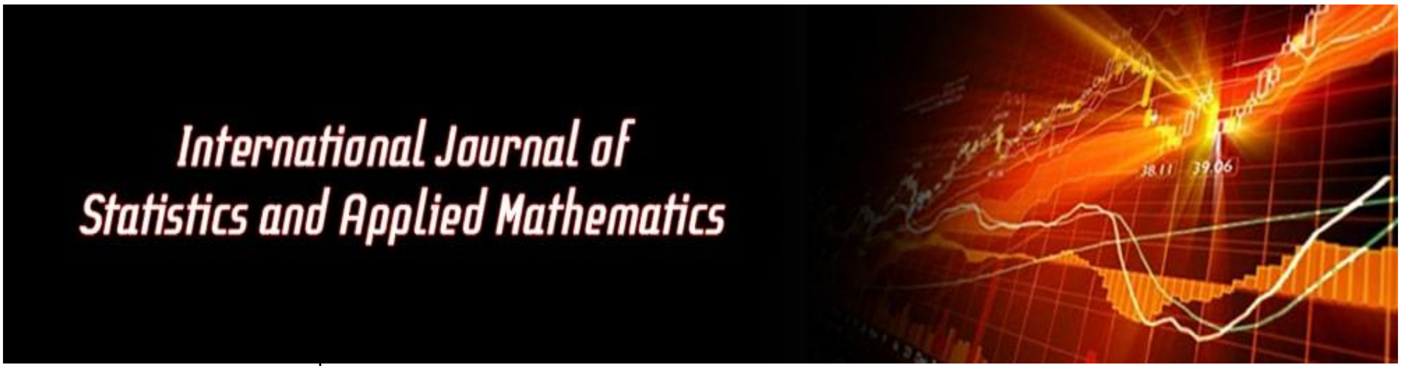

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Jyoti Pandey
Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur, Uttar Pradesh, India

Uma Srivastava
Professor, Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur, Uttar Pradesh, India

Corresponding Author:
Jyoti Pandey
Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur, Uttar Pradesh, India

Approximating the three parameters of Gumbel model through Lindley Bayes

Jyoti Pandey and Uma Srivastava

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Abstract
The three-parameter Gumbel distribution has become very useful in recent years because it can be applied in many areas such as hydrology, climate studies, weather analysis, environmental research, and finance. The distribution is characterized by three parameters shape (α), scale (β), and location (γ) which together control the skewness, spread, and central tendency of the data. Accurate estimation of these parameters plays a key role in improving statistical modelling and inference. In this study, we use a Bayesian estimation method for finding the parameters of the three-parameter Gumbel distribution. The Bayesian approach is useful because it combines prior information about the parameters with the data, which helps in getting better estimates, especially when the sample size is small. The main difficulty in Bayesian methods is that the required calculations are often very complex. To deal with this problem, we apply Lindley's approximation, which is a simple and efficient way to get approximate solutions. Using this method, we obtain the Approximate Bayes Estimators (ABEs) of the parameters α, β, and γ under the Squared Error Loss Function (SELF). The results show that ABS performs better in terms of mean squared error, especially for smaller sample sizes.

Keywords: MLE, ABS, Gumbel distribution, Bayesian estimation, Lindley's approximation, scale parameter, squared error loss function

Introduction
Bayes estimators are used to estimate the parameter of a Gumbel distribution under a Bayesian framework. For the three parameter Gumbel distribution (α, β, γ). The Bayesian estimation provides a way to incorporate prior knowledge or beliefs about the parameters in form of distribution in the estimation process. Bayesian estimation allows the use of prior knowledge or expert opinion about the parameter (α, β, γ) through prior distributions. In cases where sample size is small or data is spares, frequentist methods (e.g. maximum likelihood estimation) may yield unstable estimates.
Bayesian methods improve the estimation process by taking advantage of prior information. Bayesian estimators provide posterior distribution for the parameters, provide not just point estimates but also posterior intervals, which quantify uncertainty about the estimates.
For distributions like the three parameter Gumbel, where maximum likelihood estimation might be challenging or computationally intense. Bayes estimator minimizes the expected posterior loss, making them adaptable to different loss function (e.g. squared error loss). This makes the estimation more optimized for particular situation.
Extreme value distributions have been an important topic of research in applied probability and statistics. Kotz and Nadarajah (2000) [1] presented a detailed theoretical framework for extreme value distributions and their applications, whereas Lawless (2003) [2] emphasized statistical models for lifetime data, which are closely related to reliability and survival analysis. From a Bayesian perspective, Bernardo and Smith (2000) [3] and Robert (2007) [4] offered fundamental contributions by developing the theoretical and computational foundations of Bayesian inference. In the same direction, Lindley (1980) [5] introduced an approximation method for Bayesian analysis, which provides an effective solution for cases where posterior integrals are intractable.

Later, Tierney and Kadane (1986) [6] extended this work by proposing accurate approximations for posterior moments and marginal densities. These works provided the basis for developing Approximate Bayes Estimation methods.

In the field of reliability analysis, Sinha (1986) [7] discussed life testing approaches that further motivate the use of flexible lifetime distributions. Within this framework, several authors have studied Gumbel and generalized Gumbel distributions. Raqab and Kundu (2005) [10] discussed Bayesian estimation for the generalized Gumbel distribution, while Sharma and Kumar (2013) specifically applied Lindley's approximation for Bayesian inference in generalized Gumbel models. Kumar and Lalhita (2012) also explored Bayesian estimation in the Gumbel distribution under various loss functions.

More recently, Okumu *et al.* (2024) [9] extended the study of the three-parameter Gumbel distribution by proposing formulations and estimation procedures. Similarly, Oluwafunmilola *et al.* (2023) [8] introduced the Lindley Exponentiated Gumbel distribution and demonstrated its applicability to environmental data, highlighting the continuing relevance of Lindley's approximation in modern distribution theory.

In parallel, Srivastava and Yadav (2018) [13] made contributions to the Bayesian estimation of the generalized compound Rayleigh distribution, applying approximate Bayes techniques under different loss functions. Their later works (Yadav & Srivastava, 2018) [13] considered estimation under entropy and precautionary loss functions, further showing the versatility of Lindley's approximation in lifetime distributions.

Overall, earlier studies show that Bayesian estimation and Lindley's approximation have been applied to generalized Gumbel, compound Rayleigh, and related distributions. However, there is still room to study the three-parameter Gumbel distribution more closely. In particular, most past research has focused on generalized versions or on different loss functions, while only a few works have directly looked at the three-parameter case under squared error loss using Lindley's method. This study addresses that gap by proposing Approximate Bayes Estimators (ABEs) for the parameters (α, β, γ) of the three-parameter Gumbel distribution and comparing them with the traditional Maximum Likelihood Estimation (MLE) through simulation.

The Gumbel Distribution

The "Gumbel Distribution" is named after Emil Julius Gumbel (1891-1966), a German Mathematician and Statistician. He is considered the founder of the Gumbel distribution, which he introduced as part of his work on the extreme value theory. He developed the distribution to model the distribution of the maximum (or minimum) values of large datasets.

"The transition from a two parameter Gumbel distribution to three parameter Gumbel distribution involves adding a shape or shift (location). It is useful when the data shows skewness, asymmetry, or systematic shift that cannot be explained by just two parameters".

1. Let x be a random variable representing the data points or observations. In practical applications, it could represent extreme values such as maximum temperatures, flood levels, or financial losses. f(x) be the probability density function value at x.

$$F(x;\ \alpha,\beta,\gamma) = \begin{cases} \frac{1}{\beta}\ e^{-\left(\frac{x-\gamma-\alpha}{\beta}\right)}\ e^{-e^{-\left(\frac{x-\gamma-\alpha}{\beta}\right)}}, x > \gamma \\ 0, x \le \gamma \end{cases} \tag{1.1}$$

Where,

γ is the location parameter determines the location of the distribution.

β > 0 is the scale parameter controls the spread of the distribution.

α is the shape parameter.

For x ≤ γ the pdf is 0 because the Gumbel distribution is not defined below this threshold when using a shift form.

For x > γ the pdf follows the exponential term.

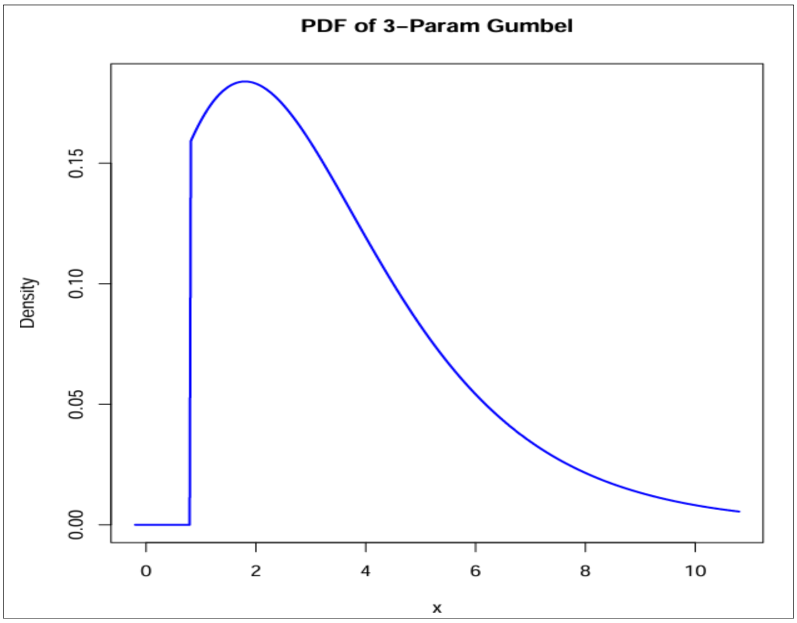

Fig 1: Probability density function of the three-parameter Gumbel distribution

2. The Cumulative distribution function for the three parameter Gumbel distribution can be computed by integrating the probability density function.

$$F(x) = e^{-e^{-\left(\frac{x-\gamma-\alpha}{\beta}\right)}} \quad (2.1)$$

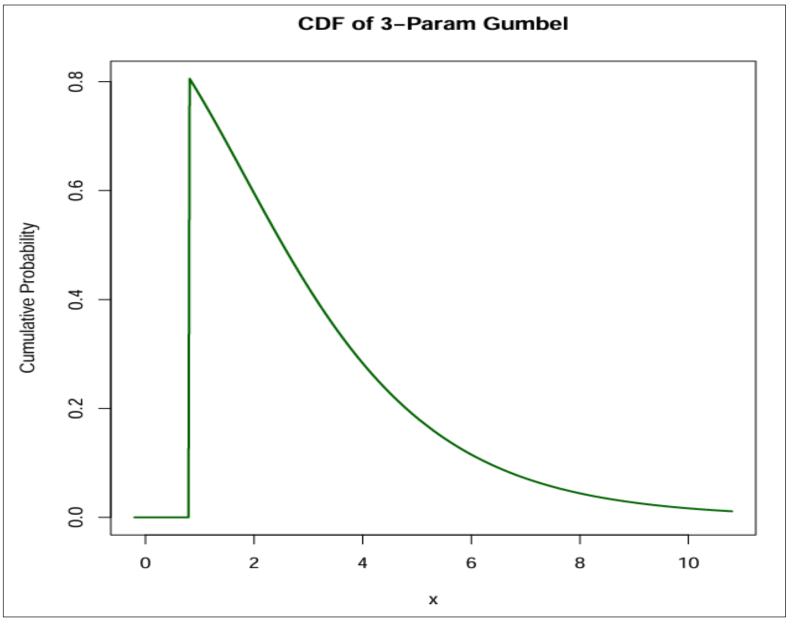

Fig 2: Cumulative distribution function of the three-parameter Gumbel distribution

3. The Maximum Likelihood Estimation for three parameter Gumbel distribution

The maximum likelihood method is the one of the best methods to estimate parameters. (α,β,γ) of the distribution, not the sample size. The sample size is $(x_1, x_2 \ldots . x_n)$ is fixed and used to calculate the likelihood of the observed data.

$L(\alpha,\beta,\gamma; \underline{x}) = \prod_{i=1}^{n} f(x_I;\ \alpha,\beta,\gamma)$

$$= \prod_{i=1}^{n} \beta e^{(-\beta(x-\gamma-\alpha))} e^{-e^{-(\beta(x-\gamma-\alpha))}} \quad (3.1)$$

$\frac{1}{\beta}$ is reparametrized as β in equation.

Simplify the likelihood function

$$L(x) = \beta e^{(-\beta(x-\gamma-\alpha))} e^{-e^{-(\beta(x-\gamma-\alpha))}} \quad (3.2)$$

Take the logarithm

$$\text{Log } L(x) = n \log \beta - \beta \sum (x_i - \gamma - \alpha) - \sum e^{-(\beta(x-\gamma-\alpha))} \quad (3.3)$$

Differentiating equation number (6.3) with respect to (α,β,γ) yield respectively we get

$$\frac{\partial \log L}{\partial \alpha} = \frac{\partial}{\partial \alpha}(n \log \beta - \beta \sum (x_i - \gamma - \alpha) - \sum e^{-(\beta(x-\gamma-\alpha))})$$

$$= \beta_n - \beta e^{-(\beta(x-\gamma-\alpha))} \quad (3.4)$$

$$\frac{\partial \log L}{\partial \beta} = \frac{\partial}{\partial \beta}\left(n \log \beta\ -\ \beta \sum (x_i - \gamma - \alpha) - \sum e^{-(\beta(x-\gamma-\alpha))}\right)$$

$$= \frac{n}{\beta} - (x_i - \gamma - \alpha) + (x_i - \gamma - \alpha) e^{-(\beta(x-\gamma-\alpha))} \quad (3.5)$$

$$\frac{\partial \log L}{\partial \gamma} = \frac{\partial}{\partial \gamma}(n \log \beta\ -\ \beta \sum (x_i - \gamma - \alpha) - \sum e^{-(\beta(x-\gamma-\alpha))}\ \beta_n - \beta e^{-(\beta(x-\gamma-\alpha))} \quad (3.6)$$

Setting the derivatives ion equation (3.4), (3.5) and (3.6) to zero and solving for (α,β,γ) gives the maximum likelihood estimators (MLEs) of the parameters: $\hat{\alpha}_{MLE}, \hat{\beta}_{MLE}$ and $\hat{\gamma}_{MLE}$. These estimates can be obtained using the Newton-Raphson method.

- **Bayes estimators of β with known parameter α and γ under Linex loss function (LLF)**

If $\hat{\alpha}$ and $\hat{\gamma}$ is known we assume (a, b) as conjugate prior for β as:-

$$G(\beta|\underline{x}) = \frac{b^a}{\Gamma a}\,\beta^{a-1}\,e^{-\beta b} \tag{4.1}$$

(a,b) >0; β >0

Combining likelihood function (3.2) and prior density (4.1) we obtain the posterior density of β in the form of

$$h(\beta|\,\underline{x}) = \frac{\prod_{i=1}^{n}[\beta\, e^{(-\beta(x-\gamma-\alpha))} e^{-e^{(-\beta(x-\gamma-\alpha))}}]\,\frac{b^a}{\Gamma a}\beta^{a-1}\,e^{-\beta b}}{\int_0^\infty \prod_{i=1}^{n}[\beta\, e^{(-\beta(x-\gamma-\alpha))} e^{-e^{(-\beta(x-\gamma-\alpha))}}]\,\frac{b^a}{\Gamma a}\beta^{a-1}\,e^{-\beta b}\,d\beta} \tag{4.2}$$

Then we combine the likelihood and the prior we obtain the posterior after solving the previous equation, we get the posterior

$$h(\beta|\,\underline{x}) = \frac{\beta^{a+n}\,e^{-\beta(u+b)}\,(u+b)^{n+a+1}}{\Gamma(n+a+1)} \tag{4.3}$$

Where, u = (x-γ-α).

5. Bayes estimate under squared error loss function (Self)

$\widehat{B}_{BS}$ is the posterior mean given by-

$$\widehat{B}_{BS} = \int_0^\infty \frac{\beta\,\beta^{a+n}\,e^{-\beta(u+b)}(u+b)^{n+a+1}}{\Gamma(n+a+1)}\,d\beta \tag{5.1}$$

$$= \frac{(u+b)^{n+a+1}}{\Gamma(n+a+1)}\int_0^\infty \beta\,\beta^{a+n}\,e^{-\beta(u+b)}\,d\beta$$

$$\widehat{B}_{BS} = \frac{a+n+1}{u+b} \tag{5.2}$$

6. Bayes estimators with unknown α,β and γ

Joint prior density α,β and γ is given by:-

$$G(\alpha,\beta,\gamma) = g_1(\alpha)g_2(\gamma)g_3(\beta|\,\gamma)$$

Taking

$$G_1(\alpha) = c \tag{6.1}$$

$$G_2(\gamma) = \frac{1}{\delta}e^{-\frac{\gamma}{\delta}} \tag{6.2}$$

$$g_3(\beta|\,\gamma) = \frac{1}{\Gamma\xi}\,\gamma^{-\xi}\beta^{\xi+1}\,e^{\left[-\frac{\beta}{\gamma}\right]} \tag{6.3}$$

$$G(\alpha,\beta,\gamma) = g_1(\alpha)g_2(\gamma)g_3(\beta|\,\gamma)$$

$$= \frac{c}{\delta\Gamma\xi}\gamma^{-\xi}\beta^{\xi+1}\,e^{\left[-\left(\frac{\gamma}{\delta}+\frac{\beta}{\gamma}\right)\right]} \tag{6.4}$$

Joint posterior with likelihood equation number (3.2) and (6.4) we get;

$$h^*(\alpha,\beta,\gamma) = \frac{\gamma^{-\xi}\beta^{\xi+1}\,e^{\left[-\left(\frac{\gamma}{\delta}+\frac{\beta}{\gamma}\right)\right]}\,L(\underline{x}|\alpha,\beta,\gamma)}{\iiint \gamma^{-\xi}\beta^{\xi+1}\,e^{\left[-\left(\frac{\gamma}{\delta}+\frac{\beta}{\gamma}\right)\right]}\,L(\underline{x}|\alpha,\beta,\gamma)\,d\alpha d\beta d\gamma} \tag{6.5}$$

The approximate Bayes estimators

$$V(\Theta) = v(\alpha,\beta,\gamma)$$

$\hat{v}_{AB} = E(v|\underline{x}) = \frac{\iiint v(\alpha,\beta,\gamma)G^*(\alpha,\beta,\gamma)d\alpha d\beta d\gamma}{\iiint G^*(\alpha,\beta,\gamma)d\alpha d\beta d\gamma}$ (6.6)

- **Lindley Approximation**

$E(v(\alpha, \beta, \gamma | \underline{x}))$

$= V(\Theta)+\frac{1}{2}[S(v_1\sigma_{11}(v_2\sigma_{12}+(v_3\sigma_{13})]+ T(v_1\sigma_{21}+v_2\sigma_{22}+v_{23}\sigma_{23})$

$+R(v_1\sigma_{31}+v_2\sigma_{32}+v_2\sigma_{33}) + v_1a_1+v_2a_2+v_3a_3+a_4+a_5+O\left(\frac{1}{n^2}\right)$ (7.1)

Evaluated at MLE = ($\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$) where;

$a_1 = \rho_1\sigma_{11}+\rho_2\sigma_{12} + \rho_3\sigma_{13}$ (7.2)

$a_2 = \rho_2\sigma_{21}+\rho_2\sigma_{22} + \rho_3\sigma_{23}$ (7.3)

$a_3 = \rho_1\sigma_{31}+\rho_2\sigma_{32} + \rho_3\sigma_{33}$ (7.4)

$a_4 = v_{12}\sigma_{12}+v_{13}\sigma_{13} + v_{23}\sigma_{23}$ (7.5)

$a_5 = \frac{1}{2}(v_{11}\sigma_{11}+v_{22}\sigma_{22} + v_{33}\sigma_{33})$ (7.6)

$S = [\sigma_{11}\ell_{111} + 2\sigma_{12}\ell_{121} + 2\sigma_{13}\ell_{131} + 2\sigma_{23}\ell_{231} + \sigma_{22}\ell_{221} + \sigma_{33}\ell_{331}]$ (7.7)

$T = [\sigma_{11}\ell_{112} + 2\sigma_{12}\ell_{122} + 2\sigma_{13}\ell_{132} + 2\sigma_{23}\ell_{232} + \sigma_{22}\ell_{222} + \sigma_{33}\ell_{332}]$ (7.8)

$R = [\sigma_{11}\ell_{113} + 2\sigma_{12}\ell_{123} + 2\sigma_{13}\ell_{133} + 2\sigma_{23}\ell_{233} + \sigma_{22}\ell_{223} + \sigma_{33}\ell_{333}]$ (7.9)

To apply Lindley approximation in the equation number (7.1)

$\Sigma_{ijk}; I, j, k = 1,2,3$

Likelihood function from equation number (3.2)

$L(x) = \beta^n e^{(-\beta(x-\gamma-\alpha))} e^{-\sum e^{(-\beta(x-\gamma-\alpha))}}$ (x;α,β,γ >0)

Log likelihood of above equation

$\text{Log } L(x) = n \log \beta - \beta \sum(x-\gamma-\alpha) - \sum e^{(-\beta(x-\gamma-\alpha))}$

Now,

$\ell_1 = \frac{\partial logl}{\partial\alpha} = \beta_n - \beta e^{-\beta(x-\gamma-\alpha)}$ (7.10)

$\ell_2 = \frac{\partial logl}{\partial\beta} = \frac{n}{\beta} - (x-\gamma-\alpha)+(x-\gamma-\alpha).\, e^{(-\beta(x-\gamma-\alpha))}$ (7.11)

$\ell_3 = \frac{\partial logl}{\partial\gamma} = \beta_n - \beta e^{-\beta(x-\gamma-\alpha)}$ (7.12)

Again,

$\ell_{11} = \frac{\partial^2 logl}{\partial\alpha^2} = -\beta^2 e^{-\beta(x-\gamma-\alpha)}$ (7.13)

$\ell_{22} = \frac{\partial^2 logl}{\partial\beta^2} = -\frac{n}{\beta^2} - (x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))}$ (7.14)

$\ell_{33} = \frac{\partial^2 logl}{\partial\gamma^2} = -\beta^2 e^{-\beta(x-\gamma-\alpha)}$ (7.15)

$\ell_{12} = \frac{\partial^2 logl}{\partial\alpha\,\partial\beta} = n - e^{-\beta(x-\gamma-\alpha)} + \beta(x-\gamma-\alpha).\, e^{(-\beta(x-\gamma-\alpha))}$ (7.16)

$$\ell_{21} = \frac{\partial^2 \log l}{\partial\beta\,\partial\alpha} = n - e^{-\beta(x-\gamma-\alpha)} + \beta(x-\gamma-\alpha).\, e^{(-\beta(x-\gamma-\alpha))} \quad (7.17)$$

From (7.16) & (7.17)

$$\ell_{12} = \ell_{21} \quad (7.18)$$

$$\ell_{13} = \frac{\partial^2 \log l}{\partial\alpha\,\partial\gamma} = -\beta^2 e^{-\beta(x-\gamma-\alpha)} \quad (7.19)$$

$$\ell_{31} = \frac{\partial^2 \log l}{\partial\gamma\,\partial\alpha} = -\beta^2 e^{-\beta(x-\gamma-\alpha)} \quad (7.20)$$

From (7.19) & (7.20)

$$\ell_{13} = \ell_{31} \quad (7.21)$$

$$\ell_{23} = \frac{\partial^2 \log l}{\partial\beta\,\partial\gamma} = 1 - e^{-\beta(x-\gamma-\alpha)} + \beta(x-\gamma-\alpha).\, e^{(-\beta(x-\gamma-\alpha))} \quad (7.22)$$

$$\ell_{32} = \frac{\partial^2 \log l}{\partial\gamma\,\partial\beta} = 1 - e^{-\beta(x-\gamma-\alpha)} + \beta(x-\gamma-\alpha).\, e^{(-\beta(x-\gamma-\alpha))} \quad (7.23)$$

From (7.22) & (7.23)

$$\ell_{23} = \ell_{32} \quad (7.24)$$

Again,

$$\ell_{111} = \frac{\partial^3 \log l}{\partial\alpha^3} = -\beta^3 e^{-\beta(x-\gamma-\alpha)} \quad (7.25)$$

$$\ell_{222} = \frac{\partial^3 \log l}{\partial\beta^3} = \frac{2n}{\beta^3} + (x-\gamma-\alpha)^{\,3} e^{(-\beta(x-\gamma-\alpha))} \quad (7.26)$$

$$\ell_{333} = \frac{\partial^3 \log l}{\partial\gamma^3} = -\beta^3 e^{-\beta(x-\gamma-\alpha)} \quad (7.27)$$

$$\ell_{112} = \frac{\partial}{\partial\alpha}\left[\frac{\partial^2 \log l}{\partial\alpha\,\partial\beta}\right] = -\beta^2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} \quad (7.28)$$

$$\ell_{113} = \frac{\partial}{\partial\alpha}\left[\frac{\partial^2 \log l}{\partial\alpha\,\partial\gamma}\right] = \beta^3 e^{-\beta(x-\gamma-\alpha)} \quad (7.29)$$

$$\ell_{121} = \frac{\partial}{\partial\alpha}\left[\frac{\partial^2 \log l}{\partial\beta\,\partial\alpha}\right] = \beta^2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} \quad (7.30)$$

$$\ell_{131} = \frac{\partial}{\partial\alpha}\left[\frac{\partial^2 \log l}{\partial\gamma\,\partial\alpha}\right] = -\beta^3 e^{-\beta(x-\gamma-\alpha)} \quad (7.31)$$

$$\ell_{221} = \frac{\partial}{\partial\beta}\left[\frac{\partial^2 \log l}{\partial\gamma\,\partial\alpha}\right] = 2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} - \beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))} \quad (7.32)$$

$$\ell_{223} = \frac{\partial}{\partial\beta}\left[\frac{\partial^2 \log l}{\partial\beta\,\partial\gamma}\right] = 2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} - \beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))} \quad (7.33)$$

$$\ell_{232} = \frac{\partial}{\partial\beta}\left[\frac{\partial^2 \log l}{\partial\gamma\,\partial\beta}\right] = 2(x-\gamma-\alpha) - \beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))} \quad (7.34)$$

$$\ell_{331} = \frac{\partial}{\partial\gamma}\left[\frac{\partial^2 \log l}{\partial\gamma\,\partial\alpha}\right] = \beta^3 e^{-\beta(x-\gamma-\alpha)} \quad (7.35)$$

$$\ell_{332} = \frac{\partial}{\partial\gamma}\left[\frac{\partial^2 \log l}{\partial\gamma\,\partial\beta}\right] = \beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))} \quad (7.36)$$

$$\ell_{231} = \frac{\partial}{\partial\beta}\left[\frac{\partial^2 \log l}{\partial\gamma\,\partial\alpha}\right] = e^{-\beta(x-\gamma-\alpha)}\left(-2\beta + \beta^2(x-\gamma-\alpha)\right) \quad (7.37)$$

$$\ell_{122} = \frac{\partial}{\partial\alpha}\left[\frac{\partial^2 \log l}{\partial\beta^2}\right] = -2(x-\gamma-\alpha)e^{(-\beta(x-\gamma-\alpha))} + \beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))} \quad (7.38)$$

$$\ell_{132} = \frac{\partial}{\partial\alpha}\left[\frac{\partial^2 \log l}{\partial\gamma\,\partial\beta}\right] = \beta^2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} \quad (7.39)$$

$$\ell_{133} = \frac{\partial}{\partial\alpha}\left[\frac{\partial^2 \log l}{\partial\gamma^2}\right] = -\beta^3 e^{-\beta(x-\gamma-\alpha)} \quad (7.40)$$

$$\ell_{233} = \frac{\partial}{\partial\beta}\left[\frac{\partial^2 \log l}{\partial\gamma^2}\right] = -e^{-\beta(x-\gamma-\alpha)}\left(-2\beta+\beta^2(x-\gamma-\alpha)\right) \quad (7.41)$$

Now,

$$-[\ell_{ijk}] = \begin{bmatrix} \ell_{111}\ell_{112}\ell_{113} \\ \ell_{221}\ell_{222}\ell_{223} \\ \ell_{331}\ell_{332}\ell_{333} \end{bmatrix}$$

From equation number (7.25) to (7.41)

$$-[\ell_{ijk}] = \begin{bmatrix} \beta^3 e^{-\beta(x-\gamma-\alpha)} & -\beta^2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} & \beta^3 e^{-\beta(x-\gamma-\alpha)} \\ 2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} - \beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))} & \frac{2n}{\beta^3} + (x-\gamma-\alpha)^3 e^{(-\beta(x-\gamma-\alpha))} & 2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} - \beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))} \\ \beta^3 e^{-\beta(x-\gamma-\alpha)} & \beta^2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} & \beta^3 e^{-\beta(x-\gamma-\alpha)} \end{bmatrix}$$

$$(7.42) = \begin{bmatrix} N_{11}N_{12}N_{13} \\ N_{21}N_{22}N_{23} \\ N_{31}N_{32}N_{33} \end{bmatrix}$$

Determinant of $-[\ell_{ijk}]$

$$D = -\left\{ \begin{matrix} N_{11}(N_{22}N_{33} - N_{23}N_{32}) + N_{12}(N_{21}N_{33} - N_{23}N_{31}) + \\ (N_{21}N_{32} - N_{31}N_{22}) \end{matrix} \right\}$$

Adjoint of matrix $= -\lfloor \ell_{ijk} \rfloor$

Cofactor of matrix $= -\lfloor \ell_{ijk} \rfloor$

$a_{11} = [N_{22}N_{33} - N_{23}N_{32}] = J_{11}$

$a_{12} = -[N_{21}N_{33} - N_{23}N_{31}]$

$= N_{23}N_{31} - N_{21}N_{33} = J_{12}$

$a_{13} = [N_{21}N_{32} - N_{22}N_{31}] = J_{13}$

$a_{21} = [N_{12}N_{33} - N_{32}N_{13}]$

$= N_{32}N_{13} - N_{12}N_{33} = J_{21}$

$a_{22} = [N_{12}N_{33} - N_{31}N_{13}] = J_{22}$

$a_{23} = [N_{11}N_{32} - N_{12}N_{31}]$

$= N_{12}N_{31} - N_{11}N_{32} = J_{23}$

$a_{31} = [N_{12}N_{23} - N_{13}N_{22}] = J_{31}$

$a_{32} = -[N_{11}N_{23} - N_{13}N_{21}]$

$= [N_{13}N_{21} - N_{11}N_{23}] = J_{32}$

$a_{33} = [N_{12}N_{22} - N_{12}N_{21}] = J_{33}$

$$[-\ell_{ijk}]^{-1} = \frac{\text{Adjoint of}[\ell_{ijk}]}{|-\ell_{ijk}|} = \begin{bmatrix} \frac{J_{11}}{D} \frac{J_{21}}{D} \frac{J_{13}}{D} \\ \frac{J_{12}}{D} \frac{J_{22}}{D} \frac{J_{23}}{D} \\ \frac{J_{13}}{D} \frac{J_{23}}{D} \frac{J_{33}}{D} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11}\sigma_{21}\sigma_{13} \\ \sigma_{12}\sigma_{22}\sigma_{23} \\ \sigma_{13}\sigma_{23}\sigma_{33} \end{bmatrix}$$

- **Approximate Bayes Estimator**

$V(\alpha, \beta, \gamma) = v$

$\widehat{V}_{AB} = E(v|\underline{x})$

Evaluated from equation number (6.4) and (7.1) joint prior density

$$G(\alpha, \beta, \gamma) = g_1(\alpha) g_2(\gamma) g_3\left(\frac{\beta}{\gamma}\right)$$

$$= \frac{c}{\partial \Gamma \xi} \gamma^{-\xi} \beta^{\xi+1} \exp\left[-\left(\frac{\gamma}{\partial} + \frac{\beta}{\gamma}\right)\right]$$

$$\rho = \log G = \log c - \log \partial - \log \Gamma\xi\,(\xi + 1) \log \beta - \xi \log \gamma - \left(\frac{\gamma}{\delta} + \frac{\beta}{\gamma}\right) \quad (8.1)$$

$$\rho_1 = \frac{\partial \rho}{\partial \alpha} = 0 \quad (8.2)$$

$$\rho_2 = \frac{\partial \rho}{\partial \beta} = \frac{\xi+1}{\beta} \frac{-1}{\gamma} \quad (8.3)$$

$$\rho_3 = \frac{\partial \rho}{\partial \gamma} = \frac{-\xi}{\gamma} \frac{-1}{\delta} + \frac{1}{\gamma^2} \quad (8.4)$$

Values of S, T and R from equation number (7.42) and from (7.28) to (7.41) we get,

$$S = [\sigma_{11}\ell_{111} + 2\sigma_{12}\ell_{121} + 2\sigma_{13}\ell_{131} + 2\sigma_{23}\ell_{231} + \sigma_{22}\ell_{221} + \sigma_{33}\ell_{331}]$$

$$= \sigma_{11}\beta^3 e^{-\beta(x-\gamma-\alpha)} + 2\sigma_{12}\beta^2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} + 2\sigma_{13}\left(-\beta^3 e^{-\beta(x-\gamma-\alpha)}\right) + 2\sigma_{23} e^{-\beta(x-\gamma-\alpha)}\left(-2\beta + \beta^2(x-\gamma-\alpha)\right)$$

$$+\sigma_{22} 2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} - \beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))} \quad +\sigma_{33}\beta^3 e^{-\beta(x-\gamma-\alpha)} \quad (8.5)$$

$$T = [\sigma_{11}\ell_{112} + 2\sigma_{12}\ell_{122} + 2\sigma_{13}\ell_{132} + 2\sigma_{23}\ell_{232} + \sigma_{22}\ell_{222} + \sigma_{33}\ell_{332}]$$

$$= \sigma_{11}\left(-\beta^2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)}\right) + 2\sigma_{12}(-2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} +$$

$$\beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))}) + 2\sigma_{13}\beta^2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)}$$

$$+2\sigma_{23} 2(x-\gamma-\alpha) - \beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))} +$$

$$\sigma_{22}\frac{2n}{\beta^3} + (x-\gamma-\alpha)\,^3 e^{(-\beta(x-\gamma-\alpha))} + \sigma_{33}\beta^2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} \quad (8.6)$$

$$R = [\sigma_{11}\ell_{113} + 2\sigma_{12}\ell_{123} + 2\sigma_{13}\ell_{133} + 2\sigma_{23}\ell_{233} + \sigma_{22}\ell_{223} + \sigma_{33}\ell_{333}]$$

$$= \sigma_{11}\beta^3 e^{-\beta(x-\gamma-\alpha)} + 2\sigma_{12}\beta^2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} +$$

$$2\sigma_{13}(-\beta^3 e^{-\beta(x-\gamma-\alpha)}) + 2\sigma_{23} e^{-\beta(x-\gamma-\alpha)}\left(-2\beta + \beta^2(x-\gamma-\alpha)\right)$$

$$+\sigma_{22} 2(x-\gamma-\alpha)e^{-\beta(x-\gamma-\alpha)} - \beta(x-\gamma-\alpha)^2 e^{(-\beta(x-\gamma-\alpha))} + \sigma_{33}\beta^3 e^{-\beta(x-\gamma-\alpha)} \quad (8.7)$$

$$\widehat{V}_{AB} = E(v|\underline{x}) = v + (v_1\alpha_1 + v_2\alpha_2 + v_3\alpha_3 + \alpha_4 + \alpha_5) + \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + R\sigma_{31})v_1 + (S\sigma_{12} + T\sigma_{22} + R\sigma_{32})v_2$$
$$+(S\sigma_{13} + T\sigma_{23} + R\sigma_{33})v_3]$$

$$= v + \phi_1 + \phi_2 \quad (8.8)$$

Were,

$$\phi_1 = (v_1 a_1 + v_2 a_2 + v_3 a_3 + a_4 + a_5) \quad (8.9)$$

$$\phi_2 = \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + R\sigma_{31})v_1 + (S\sigma_{12} + T\sigma_{22} + R\sigma_{32})v_2$$

$$+(S\sigma_{13} + T\sigma_{23} + R\sigma_{33})v_3] \ (8.10)$$

Evaluated at the MLE $\hat{V} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ where;

$$a_1 = \rho_1\sigma_{11} + \rho_2\sigma_{12} + \rho_3\sigma_{13}$$

$$= o.\sigma_{11} + \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right]\sigma_{12} + \left[\frac{-\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right]\sigma_{13} \quad (8.11)$$

$$a_2 = \rho_1\sigma_{21} + \rho_2\sigma_{22} + \rho_3\sigma_{23}$$

$$= \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right]\sigma_{22} + \left[\frac{-\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right]\sigma_{23} \ (8.12)$$

$$a_3 = \rho_1\sigma_{31} + \rho_2\sigma_{32} + \rho_3\sigma_{33}$$

$$= \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right]\sigma_{32} + \left[\frac{-\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right]\sigma_{33} \quad (8.13)$$

$$a_4 = v_{12}\sigma_{12} + v_{13}\sigma_{13} + v_{23}\sigma_{23} \quad (8.14)$$

$$a_5 = \frac{1}{2}(v_{11}\sigma_{11} + v_{22}\sigma_{22} + v_{33}\sigma_{33}) \quad (8.15)$$

- **Approximate Bayes Estimate under Squared error loss function (SELF)**

$$\hat{V}_{ABS} = E(v|\underline{x})$$

$$= v + (v_1 a_1 + v_2 a_2 + v_3 a_3 + a_4 + a_5) + \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + R\sigma_{31})v_1 + (S\sigma_{12} + T\sigma_{22} + R\sigma_{32})v_2 + 0\left(\frac{1}{n^2}\right) \quad (9.1)$$

$$v = \varphi_1 + \varphi_2 \quad (9.2)$$

Approximate Bayes Estimate

$$\hat{V}_{ABS} = E(\theta|\underline{x}) = \theta \quad (9.3)$$

$$E(\theta|\underline{x}) = \frac{\iiint \theta G^*(\alpha,\beta,\gamma)\, d\alpha d\beta d\gamma}{\iiint G^*(\alpha,\beta,\gamma)\, d\alpha d\beta d\gamma} \quad (9.4)$$

The above equation (9.4) is evaluated by method of Lindley Approximation, whose simplified form is equation number (9.1) replace θ by $V_{\alpha\beta\gamma}$ in equation number (9.4) and (9.1).

Special cases:-

- **Approximate Bayes estimate of α**

$$V_{\alpha\beta\gamma} = v = \alpha$$

$$v_1 = 1; v_{11} = v_{12} = v_{13} = 0$$

$$v_2 = v_{21} = v_{22} = v_{23} = 0$$

$$v_3 = v_{31} = v_{32} = v_{33} = 0$$

$$E(v|\underline{x}) = \alpha + \varphi_1 + \varphi_2 \quad (9.5)$$

$$\varphi_1 = v_1 a_1 + v_2 a_2 + v_3 a_3 + a_4 + a_5 \quad (9.6)$$

$= 1\left\{0\sigma_{11} + \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right] \sigma_{12} + \left[-\frac{\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right] \sigma_{13}\right\}$ (9.6) $\varphi_2 = \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + R\sigma_{31})]$ (9.7)

$$E(\alpha|\underline{x}) = \alpha + \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right] \sigma_{12} + \left[-\frac{\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right] \sigma_{13} + \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + R\sigma_{31})] \quad (9.8)$$

$$= \alpha + \varphi'$$

$$\widehat{\alpha}_{ABS} = \alpha + \varphi' \quad (9.9)$$

Were,

$$\varphi' = \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right] \sigma_{12} + \left[-\frac{\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right] \sigma_{13} + \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + R\sigma_{31})]$$

- **Approximate Bayes estimate of β**

$$V_{\alpha\beta\gamma} = v = \beta$$

$$v_2 = 1; v_{21} = v_{22} = v_{23} = 0$$

$$v_1 = v_{11} = v_{12} = v_{13} = 0$$

$$v_3 = v_{31} = v_{32} = v_{33} = 0$$

$$E(v|\underline{x}) = \beta + \varphi_1 + \varphi_2 \quad (9.10)$$

$$\varphi_1 = \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right] \sigma_{22} + \left[-\frac{\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right] \sigma_{23} \quad (9.11)$$

$$\varphi_2 = \frac{1}{2}[(S\sigma_{12} + T\sigma_{22} + R\sigma_{32})] \quad (9.12)$$

$$E(\beta|\underline{x}) = \beta + \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right] \sigma_{22} + \left[-\frac{\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right] \sigma_{23}$$

$$+\frac{1}{2}[(S\sigma_{12} + T\sigma_{22} + R\sigma_{32})] \quad (9.13)$$

$$\widehat{\beta}_{ABS} = \beta + \varphi'' \quad (9.14)$$

$$\varphi'' = \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right] \sigma_{22} + \left[-\frac{\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right] \sigma_{23} + \frac{1}{2}[(S\sigma_{12} + T\sigma_{22} + R\sigma_{32})] \quad (9.15)$$

- **Approximate Bayes estimate of γ**

$$V_{\alpha\beta\gamma} = v = \gamma$$

$$v_3 = 1; v_{31} = v_{32} = v_{33} = 0$$

$$v_1 = v_{11} = v_{12} = v_{13} = 0$$

$$v_2 = v_{21} = v_{22} = v_{23} = 0$$

$$E(v|\underline{x}) = \gamma + \varphi_1 + \varphi_2 \quad (9.16)$$

$$\varphi_1 = \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right] \sigma_{32} + \left[-\frac{\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right] \sigma_{33} \quad (9.17)$$

$$\varphi_2 = \frac{1}{2}[(S\sigma_{13} + T\sigma_{23} + R\sigma_{33})] \quad (9.18)$$

$$E(\gamma|\underline{x}) = \beta + \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right] \sigma_{32} + \left[-\frac{\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right] \sigma_{33} + \frac{1}{2}[(S\sigma_{13} + T\sigma_{23} + R\sigma_{33})] \quad (9.19)$$

$$\widehat{\gamma}_{ABS} = \gamma + \varphi''' \quad (9.20)$$

$$\varphi''' = \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right] \sigma_{32} + \left[-\frac{\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right] \sigma_{33} + \frac{1}{2}[(S\sigma_{13} + T\sigma_{23} + R\sigma_{33})] \quad (9.21)$$

- **Simulation Design**

Table 1: Parameters and priors used in simulation study

Parameter Symbol	Description	Value Used	Role
α	Shape parameter	1	Determines the skewness/shape
β	Scale parameter	0.5	Controls spread (higher = more spread)
γ	Location parameter	1.5	Distribution starts after this
ξ	Hyperparameter in prior for β	5	Used in prior: $g_3(\beta)$
δ	Hyperparameter in prior for γ	1.5	Used in prior: $g_2(\gamma)$
n	Sample size	Sample size	Varying in simulation
Repetitions	Number of simulations runs per sample size	1000	To reduce fluctuation
Simulation Goal	Estimate α, β, γ using MLE and ABS (Lindley)		With low fluctuation
Distribution	3-parameter Gumbel (shifted version)		PDF defined only for x > γ

The focuses on estimating the parameters of the Gumbel Three-Parameter Distribution, which includes the shape parameter α, the scale parameter β, and the location parameter γ. Two methods of estimation were considered: Maximum Likelihood Estimation (MLE) and Approximate Bayes Estimation (ABS). The comparison of these methods was based on simulation studies for different sample sizes.
To evaluate and compare MLE and ABS, simulation experiments were performed for various sample sizes:
n=10, 20, 30, 40, 50, 60, 70, 80
For each sample size, 500 datasets were generated from the Gumbel distribution with fixed parameter values α=1, β=0.5, and γ=1.5 (refer to Equation 6.2 to 6.6). Both MLE and ABS estimates were calculated for each simulated dataset

- In each iteration, data were generated from the Gumbel distribution with fixed true values of parameters.
- Both MLE and ABS estimates for α, β, γ were computed for each generated dataset.

Estimation Methods

- **Maximum Likelihood Estimation (MLE):** This classical method estimates parameters by maximizing the likelihood function based on the observed data.
- **Approximate Bayes Estimation (ABS):** This method incorporates prior information and uses approximations (like Lindley's method) to obtain Bayesian estimates of the parameters under the Squared Error Loss Function.
- **Source of the Table:** The table is generated using the Lindley Approximation method for Approximate Bayes Estimation (ABS) because The ABS values are computed using prior distributions and the Lindley approximation formula.

Each row of the table corresponds to a sample size n, ranging from 10 to 80.

Each column gives:-
MLE of α, β, γ, along with their Mean Squared Error (MSE) in square brackets.
Bayesian Estimator (ABS) of α, β, γ (using Lindley approximation), again with MSEs in brackets.

Mean and MSE'S of α, β, γ

α=1, β=0.5, and γ=1.5

Table 2: Mean estimates and MSEs of α, β, and γ under MLE and ABS

n	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{ABS}$	$\hat{\beta}_{MLE}$	$\hat{\beta}_{ABS}$	$\hat{\gamma}_{MLE}$	$\hat{\gamma}_{ABS}$
10	1.1719762 [0.010896597]	1.1775973 [0.005448299]	1.5953965 [0.011724196]	1.6346384 [0.005862098]	1.3950228 [0.3206442]	1.3597961 [0.3405968]
20	1.4640277 [0.040077145]	1.5025884 [0.020038573]	1.6654546 [0.011068272]	1.6795053 [0.005534136]	0.9466313 [0.4909020]	0.9343685 [0.5073344]
30	1.6893457 [0.073350322]	1.7410376 [0.036675161]	1.8099570 [0.021586539]	1.7815978 [0.010793270]	0.7536863 [0.6731001]	0.7448584 [0.6803468]
40	1.7457150 [0.015337312]	1.7341551 [0.007668656]	1.8982915 [0.113813694]	1.9281840 [0.056906847]	0.6598134 [0.7967643]	0.6557527 [0.8020917]
50	2.0213232 [0.015139596]	2.0833361 [0.007569798]	1.9940986 [0.032979120]	2.0253375 [0.016489560]	0.5651210 [0.9483692]	0.5615013 [0.9531808]
60	2.1614104 [0.126539635]	2.1950215 [0.063269818]	2.1057310 [0.216895597]	2.0974956 [0.108447798]	0.5050871 [1.0511569]	0.5059727 [1.0490784]
70	2.3438446 [0.002884391]	2.4003034 [0.001442195]	2.1919423 [0.043212979]	2.1513018 [0.021606490]	0.4398934 [1.1653871]	0.4404160 [1.1635045]
80	2.6614196 [0.058461375]	2.6216654 [0.029230687]	2.3055205 [0.012385424]	2.2615925 [0.006192712]	0.4300852 [1.1868487]	0.4323566 [1.1814668]

Analysis of Estimation
The parameter estimation for the three-parameter distribution was carried out using both the Maximum Likelihood Estimator (MLE) and the Approximate Bayesian Estimator (ABS), obtained using Lindley's approximation. The estimations were performed for the parameters α, β, and γ across a range of sample sizes from 20 to 80. To evaluate the effectiveness of both estimators,

repeated simulations were performed, and smoothed graphical representations were created for each parameter to represent how the estimators behaved with increasing sample size.

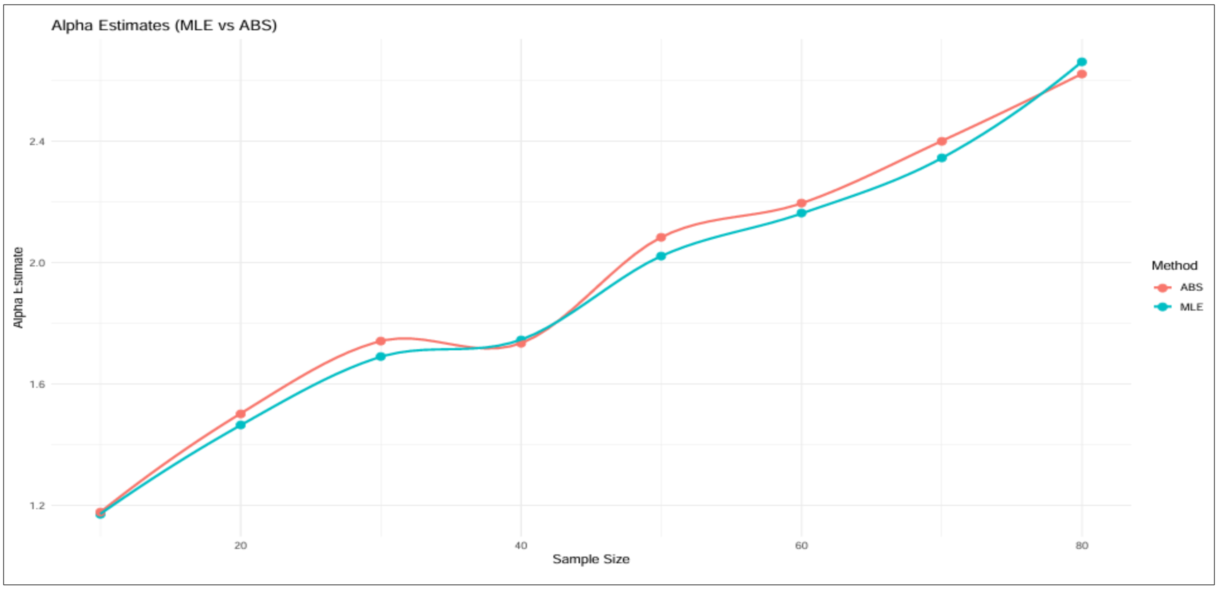

Fig 3: Behaviour of parameter estimates under MLE and ABS methods

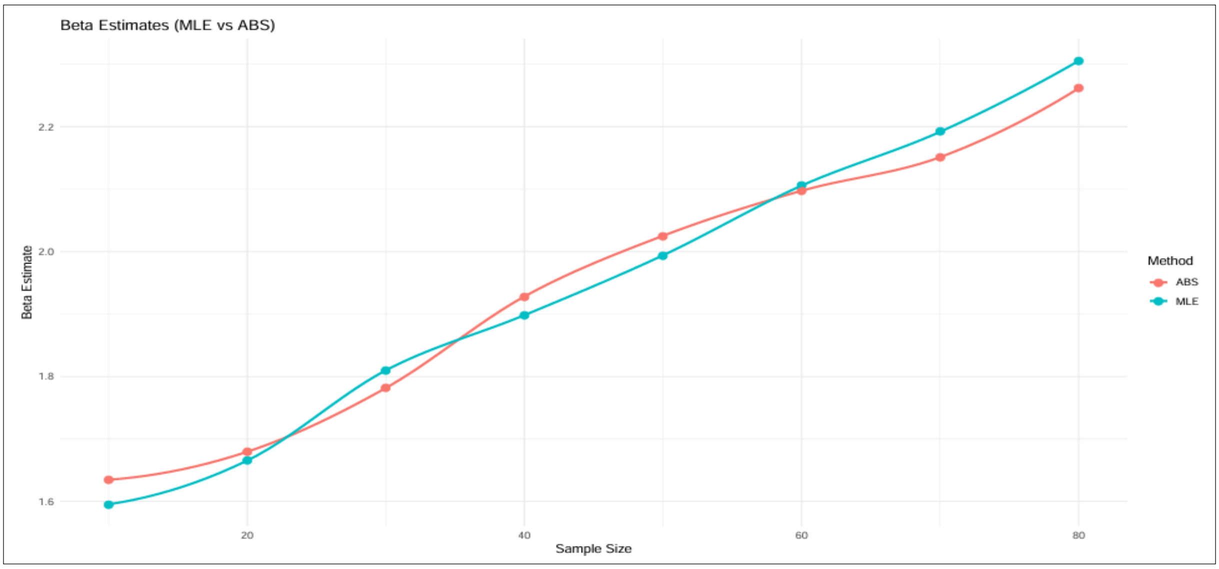

Fig 4: Estimation performance of parameter β under MLE and ABS

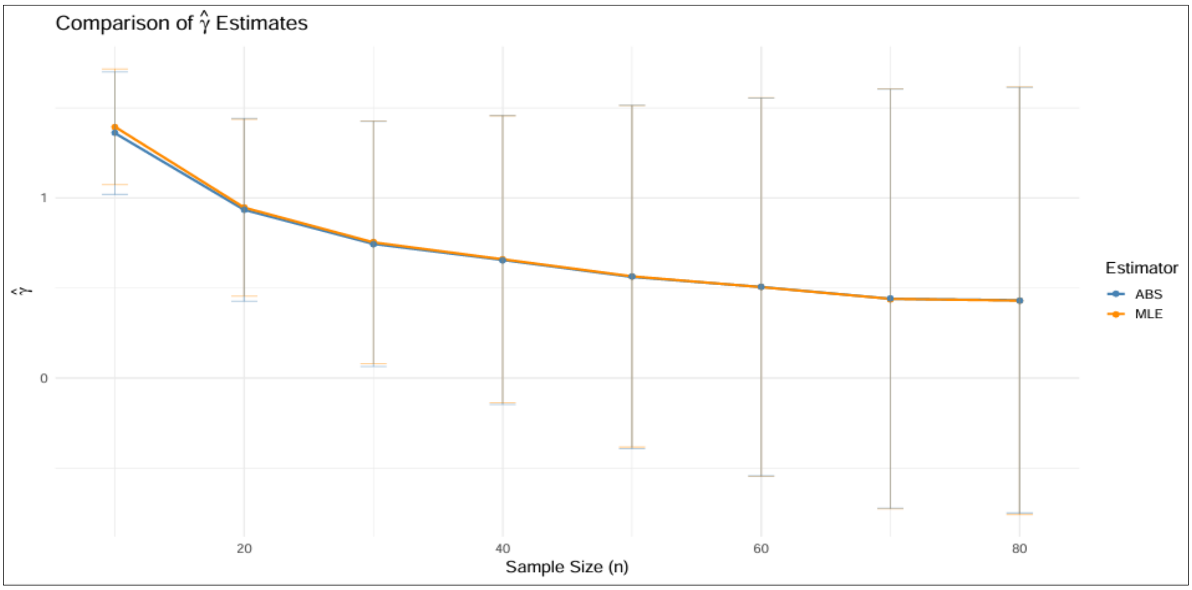

Fig 5: Estimation performance of parameter γ under MLE and ABS

Analysis of Alpha Estimates

The first chart displays the estimation behaviour of the parameter α under both the MLE and ABS methods. At smaller sample sizes (N=20, 30), both estimators show some variation, particularly at smaller sample sizes. However, the Approximate Bayesian Estimator (ABS) generally provides greater stability and produces estimates that are closer to the true parameter values than those from the Maximum Likelihood Estimator (MLE). As the sample size increases beyond 40, both estimators progressively become more reliable with their estimates approaching the actual parameter values indicating that both are consistent.

Analysis of Beta Estimates

The second chart presents the estimation results for the parameter β. Here, both the MLE and ABS estimators perform very well across all sample sizes. While small fluctuations are observed at lower sample sizes, these are less significant than the α estimates. Both estimators converge rapidly to the true parameter value as the sample size increases, indicating high efficiency and consistency. However, the ABS estimator continues to show a marginal improvement in stability for smaller sample sizes, highlighting its reliability in real world situations with limited data.

Analysis of Gamma Estimates

The third chart displays the behaviour of the gamma (γ) estimates obtained through both the Maximum Likelihood Estimator (MLE) and the Approximate Bayesian Estimator (ABS) across increasing sample sizes. A clear characteristic of the plot is the decreasing trend in the estimates with increasing sample size, which is largely due to the increased sensitivity in estimating the location parameter (γ) at smaller sample sizes. The γ parameter represents the minimum or threshold value in the distribution. In small samples, there is a higher likelihood that the true minimum value is not captured due to limited data, which causes the estimator to overestimate γ. As a result, for smaller sample sizes, both MLE and ABS initially produce gamma estimates that are higher than the true parameter value, showing a positive bias.

As the sample size increases, the probability of observing values closer to the true minimum rises. This additional information allows both estimators to correct the earlier overestimation, causing the gamma estimates to gradually decrease and converge towards the true value. The Approximate Bayesian Estimator shows a smoother and more consistent decrease than MLE, mainly because prior knowledge in the Bayesian method helps stabilize estimates when data is limited. Thus, the downward trend in the gamma estimates reflects the correction of small-sample bias in location parameter estimation as more information becomes available with increasing sample sizes. This behaviour is well-known in reliability analysis and extreme value modelling, where location parameters are highly sensitive to sample extremes.

Conclusion

The study shows that both Maximum Likelihood Estimator (MLE) and Approximate Bayesian Estimator (ABS) perform well in estimating the parameters α, β, and γ of the three-parameter distribution, especially as the sample size increases. However, the ABS estimator generally provides more stable and accurate results for smaller sample sizes due to its ability to include prior information. For α, ABS reduces variability and bias better than MLE in small samples. For β, both estimators perform almost equally well, with ABS having a slight advantage in stability. For γ, both estimators initially overestimate due to the small sample size, but ABS adjusts more smoothly as more data becomes available. "Overall, the Approximate Bayesian Estimator performs better with smaller sample sizes, whereas both methods yield comparable outcomes for larger samples, indicating stable and accurate estimation performance.

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