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Scale mixtures of Normals with Rayleigh priors in Tobit quantile regression

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Abstract
This study introduces a new hierarchical formulation of the Bayesian Lasso by incorporating the Scale Mixture of Normals mixing with Rayleigh (BSCNRMIXING prior) into the Tobit Quantile Regression (Tobit Q Regression) framework. The BSCNRMIXING prior is proposed as a promising alternative to the widely used Scale Mixture of Normals mixing with Rayleigh (BSCNRMIXING prior), providing enhanced effectiveness in achieving simultaneous coefficient estimation and variable selection within the Bayesian Lasso paradigm. For Bayesian inference, Gibbs sampling schemes are derived for the full conditional posterior distributions. The proposed methodology is rigorously examined through comprehensive simulation experiments and an application to real data, with comparative analyses against established approaches, thereby highlighting its efficiency, stability, and robustness.

Keywords: Variable selection, BSCNRMIXING prior distribution, lasso, new prior

1. Introduction

Tobit Quantile regression (Tobit Q Regression) models have gained substantial prominence since the pioneering work of Powell (1986) [22], and have since been widely applied across diverse fields, including Medical Expenditures (Yue and Hong (2012) [28], Female Labor Supply (Cunha, Divino, and Saulo (2021) [8], Education Outcomes in Iraqi Secondary Schools (Salih, Majid, and Muhsen (2024) [23], Clinical Trials (Wang, Z., *et al* (2023) [25], among others. In contrast to traditional tobit regression approaches, tobit Q Regression belongs to a robust class of models (Alhusseini, F. H. H. (2017) [4]. Notably, quantile regression does not require distributional assumptions on the error term, thereby offering greater statistical efficiency than standard mean regression, particularly in the presence of non-normal error distributions. In Tobit Q regression models, the inclusion of a large number of explanatory variables is common. These covariates may exhibit varying degrees of association with the censored response variable, while others may contribute little to no explanatory power within the model Ji, Y., Lin, N., & Zhang, B. (2012) [11]. Therefore, the inclusion of weak explanatory variables in the model lacks justification, as their presence may lead to inefficiency in estimation, inflated standard errors, and reduced interpretability of the results. However, identifying and diagnosing such weak independent variables remains a highly challenging task, particularly in high-dimensional settings or when multicollinearity is present. This challenge underscores the importance of employing robust statistical techniques such as variable selection procedures (Tibshirani, R. (1996) [24], penalization methods (e.g., Lasso and Elastic Net) (Tibshirani, R. (1996) [24], (Zou and Hastie, (2005) [29], or Bayesian shrinkage approaches to ensure that only the most informative covariates are retained in the model. This procedure demonstrates high effectiveness in constructing regression models, as it possesses a strong capability to select relevant explanatory variables while excluding those with limited or no explanatory power (Fan, J., & Lv, J. (2010) [9]. Recently, many researchers have employed efficient regularization methods in combination with the Bayesian approach. Park and Casella (2008) [19] introduced the Bayesian Lasso within the framework of traditional regression models. These ideas were later extended to Tobit quantile regression. For instance, Alhamzawi (2013) [1] proposed the use of an adaptive Lasso approach in Tobit quantile regression through a Bayesian framework.

Subsequently, Alhamzawi and Yu (2014) [3] developed a Bayesian method for coefficient estimation in the Tobit quantile regression model by employing a g-prior distribution with a ridge parameter. Moreover, Alhamzawi (2014) [2] introduced a Bayesian elastic net penalty for the Tobit quantile regression model. In the field of penalized Bayesian Tobit quantile regression, most methods initially relied on the scale mixture of normal (SMN) distribution before the development and application of the Bayesian Lasso in regression models (Alhusseini, F. H. H., & Georgescu, V. (2018) [4, 5]. Flaih, A. N., *et al* (2020) introduced an innovative approach to implement the Bayesian Lasso in conventional regression models by utilizing a Scale Mixture of Normals mixing with Rayleigh (BSCNRMIXING prior) distributions to represent the Laplace density. This formulation allows for more flexible prior specification and facilitates efficient posterior computation, thereby enhancing the performance of Bayesian Lasso in estimating regression coefficients and performing variable selection. In this study, we propose a novel Bayesian Lasso approach for Tobit quantile regression by employing a Scale Mixture of Normals mixing with Rayleigh (BSCNRMIXING prior) distributions. Based on the resulting posterior distributions, we develop a tractable and computationally efficient algorithm using Markov Chain Monte Carlo (MCMC) techniques. This framework facilitates accurate estimation of regression coefficients and effective variable selection in the presence of censoring.

2.Tobit quantile regression

Tobit quantile regression (often called censored quantile regression (CQR), at zero point) models a latent outcome

$$y_i = \max(y_i^*, 0)\ , \text{where } y_i^* = x_i^T \beta_\tau + u_i \tag{1}$$

where $x_i \in R^p$, denotes regressors, β_τ, is the vector of coefficients associated with the conditional $\tau - quantile$, and $u_i$is an unobserved error term. The quantile restriction is imposed by requiring the τ_{th}

$$Q_\tau(u_i|x_i) = 0 \Rightarrow Q_\tau(y_i^*|x_i) = x_i^T \beta_\tau \tag{2}$$

This formulation extends the classical (uncensored) quantile regression framework of Koenker & Bassett (1978) [15] to cases where the response is censored. A natural estimator is obtained by minimizing the quantile (check) loss applied to the observed, censored residuals. With the check loss function $\rho_\tau(u) = u(\tau - I\{u < 0\})$. Powell (1986) [22] 1989 introduce estimator solves of CQR as following:

$$\hat{\beta}_\tau = argmin \sum_{i=1}^{n} \rho\tau(y_i - max\{0, x_i^T \beta_\tau + u_i\} \tag{3}$$

The estimation of parameters in the Tobit Quantile Regression (T Q Regression) model is performed through the minimization of the objective function specified in equation [3]. A significant computational difficulty, however, arises from the fact that equation [3] is non-differentiable at zero, which restricts the use of conventional gradient-based optimization procedures. To address this limitation, Koenker and D'Orey (1987) [14] introduced an effective approach utilizing linear programming methods. (T Q Regression) model have been extensively investigated and several estimation algorithms have been proposed, many of these methods become computationally inefficient when the proportion of censored observations is high. At present, coefficient estimation for Tobit QR can be implemented through the crq function available in the quantreg package (Koenker, 2011).

3. Bayesian Tobit Quantile Regression

Bayesian Tobit quantile regression combines the Tobit framework for censored outcomes with Bayesian quantile regression to estimate conditional quantiles of a latent (uncensored) response while fully accounting for censoring uncertainty. The Bayesian approach typically models the regression errors with an asymmetric Laplace distribution (ALD). It take the following formula:

$$f(u_i|\sigma, \tau) = \tau(1-\tau)\exp(-\rho_\tau\{(\epsilon_i)\}\ )\ \ (\text{with } \mu = 0 \text{ and } \sigma = 1\ ) \tag{4}$$

With mean, $E(u_i) = \frac{1-2\tau}{\tau(1-\tau)}$ and variance, var $(u_i) = \frac{1-2\tau+2\tau^2}{\tau^2(1-\tau)^2}$.

From these information, The joint distribution of $y = (y_1, y_2, \dots y_n)^T\ given\ X = (x_1, x_2, \dots x_n)^T$, is:

$$(y|X, \beta, \sigma, \tau) = \tau^n(1-\tau)^n exp\{-\sum_{i=1}^{n} \rho\tau(y_i - max\{0, x_i^T \beta_\tau + u_i\}\} \tag{5}$$

Maximizing the likelihood function in equation [5] is equivalent to minimizing the expression in equation [3]. However, directly employing the Laplace distribution results in complex computations and a highly challenging algorithm implementation. However, Kozumi and Kobayashi (2011) [17] demonstrated that the asymmetric Laplace distribution (ALD) can be reformulated as a function of the scale mixture of normal (SMN) distributions. Therefore, the likelihood function in equation [5] can be alternatively expressed as follows:

$$y_{i=max\{0, y_i^*\}},$$

$$y_i^*|, \beta_\tau, m_i \sim N(\alpha_\tau + x_i^T \beta_\tau + (1-2\tau)z_i, 2z_i) \tag{6}$$

A convenient working likelihood because its mode corresponds to the targeted conditional quantile and then uses latent-variable representations of the ALD to obtain tractable Gibbs or other MCMC samplers for posterior inference (Yu & Moyeed, 2001; Yu

& Stander, 2007; Kozumi & Kobayashi, 2011) [26, 27, 17]. This formulation permits straightforward incorporation of prior information, hierarchical extensions (e.g., random effects or shrinkage priors), and posterior measures of uncertainty that are often difficult to obtain with classical censored-quantile methods (Yu & Stander, 2007; Kozumi & Kobayashi, 2011) [27, 17]. Practical implementations exploit the ALD's mixture-of-normals representation to derive full conditional distributions and efficient Gibbs updates, which markedly improve computational performance versus naïve Metropolis schemes (Kozumi & Kobayashi, 2011) [17]. Recent Bayesian work has extended the Tobit-quantile framework to handle endogeneity, panel/longitudinal structures, and variable selection, showing that Bayesian methods can flexibly address identification and model-selection issues that arise with heavy censoring (Kobayashi, 2015; Ji, Lin, & Zhang, 2012) [11]. The Bayesian Tobit-QR literature builds on foundational frequentist developments for censored quantile estimation (Powell, 1986; Koenker & D'Orey, 1987; Chernozhukov & Hong, 2002; Portnoy, 2003; Peng & Huang, 2008) [22, 14, 7, 21, 20] and is supported by software implementations and packages (e.g., quantreg, bayesQR) that facilitate applied use and comparison with classical estimators (Koenker, 2005; Koenker *et wal*., CRAN) [13]. Overall, Bayesian Tobit quantile regression offers a conceptually coherent and computationally practical route to estimating conditional quantiles under censoring while allowing full posterior inference, flexible prior modeling, and extensions for endogeneity and hierarchical data structures (Yu & Stander, 2007; Kozumi & Kobayashi, 2011; Kobayashi, 2015) [27, 17]. Then the likelihood function of the probability density function $f(y_i^*|,\beta_\tau,z_i)$ is

$$f(y_i^*|,x_i^T,\tau,\beta_\tau,z_i)=\left[\frac{1}{\sqrt{4\pi z_i}}\right]^n e^{-\sum_1^n\frac{\left(y_i^*-x_i^T\beta_\tau-(1-2\tau)z_i\right)^2}{4z_i}} \quad (7)$$

The probability density function presented above (6) is considered one of the most important functions for estimating the parameters of the Tobit quantile regression model within the Bayesian framework.

3.1 A New Laplace distribution

The Laplace prior distribution (double-exponential) is widely used in Bayesian inference for sparse modeling and variable selection. Its sharp peak at zero and heavy tails induce shrinkage on regression coefficients, driving many toward zero while allowing important ones to remain large. This balances bias and variance effectively, making it a powerful alternative to Gaussian priors, particularly in the context of the Bayesian lasso (Tibshirani, 1996; Park & Casella, 2008) [24, 19]. The Laplace have the probability density function (PDF) is defined as

$$f(\beta_j|\lambda)={}^{\lambda}/_{2}\ e^{-\lambda|\beta_j|} \quad (8)$$

Where λ is the shrinkage parameter and $(\lambda\geq 0)$.

Direct utilization of the Laplace distribution in Bayesian inference often leads to substantial computational burdens and inefficiencies in estimation procedures. To address these challenges, various alternative formulations of the Laplace prior have been developed, enabling more efficient computation while preserving its favorable shrinkage characteristics see (Andrews and Mallows (1974), (Alhusseini, F. H. H. (2017) [4]. In this paper, we introduce a new hierarchical model representation by expressing the double-exponential (Laplace) prior distribution for the parameters as a scale mixture of normal distributions with the mixing density specified by the Rayleigh distribution with tobit quantile regression.

3.2 Scale Mixture of Normal Distribution

We introduce a hierarchical model in which the double-exponential prior for the parameters is reformulated as a scale mixture of normal distributions, with a Rayleigh mixing distribution.

$$\frac{1}{2b}e^{\left(\frac{|z|}{b}\right)}=\int_0^\infty\frac{1}{\sqrt{2\pi s^2}}\exp\left(\frac{-z^2}{2s^2}\right)\ \frac{s}{b}e^{\left(\frac{-s^2}{2b}\right)}ds \quad (9)$$

Then, $z\sim Laplace\ distribution\ with\ mean\ zero\ and\ b\ prameter, z|s\sim N(0,s^2)$ and $s\sim Rayleigh(b)$. We specify a zero-mean normal prior distribution for β

Let $b=\frac{\sigma^2}{\lambda}, =\beta, s=\sigma\sqrt{\tau}$, as the result the (8) is become

$$\frac{\lambda}{2\sigma^2}e^{\left(\frac{\lambda|\beta|}{\sigma^2}\right)}=\int_0^\infty\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(\frac{-\beta^2}{2\sigma^2\tau}\right)\ \frac{\lambda}{2}e^{\left(\frac{-\lambda\tau}{2}\right)}d\tau \quad (10)$$

We specify a zero-mean normal prior distribution for β taking form of

$$,f(\beta|\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(\frac{-\beta^2}{2\sigma^2\tau}\right) \quad (11)$$

Where σ^2 is = unknown prior variance of the parameter β, Then we assign an Rayleigh prior distribution on τ is takes the following form

$$\frac{\lambda}{2}\ e^{\left(\frac{-\lambda\tau}{2}\right)} \quad (12)$$

The parameter σ^2 give as guarantees unimodal posterior distribution (T. Park, and, G. Casella(2008) [19]. In summary, the proposed Bayesian hierarchical model can be expressed as:

$y_{i=max\{0,y_i^*\},i=1,....,n,,}$

$y_i^*|,\beta_\tau,x_i \sim N(x_i^T\beta_\tau+(1-2\tau)z_i,2z_i),$

$\beta|\sigma^2 A_\tau \tau_1,\tau_2 \ldots \tau_p \sim N_P(0,\sigma^2 A_\tau)$

$A_\tau = \text{diag}(\tau_1,\tau_2,,,,,,\tau_p)$ $\tau|\lambda \sim \text{Rayleigh}(\lambda)$

$\tau \sim \tau^{a-1}\exp(-b\tau)$

$\lambda \sim \lambda^{c-1}\exp(-d\lambda)$

$\sigma^2,\tau_1,\tau_2,\ldots.\tau_p \sim \pi(\sigma^2)\ d(\sigma^2)\prod_{i=1}^{p}\frac{\lambda}{2}e^{\frac{\lambda\tau_j}{2}}\ d\tau_j$

$a,b,c\ and\ d$ are hyperparameters (Alhusseini, F. H. H., & Al-Naeli, A. A. J. (2024).

3.3 Conditional Posterior Distribution Computation Inferences

3.3.1-Updating conditional posterior of y_i is

$$y_i|y_i^*,x_i,m_i,\alpha_\tau,\beta_\tau \sim \begin{cases} \gamma(y_i)\ if\ y_i\ > o \\ N(x_i^T\beta_\tau+(1-2\tau)z_i,2z)I(y_i\le 0)\ \text{otherwise}\end{cases}$$

Where $\gamma(y_i)$ denotes a degenerate distribution placing mass at the observed uncensored value y_i is normal distribution with mean $(x_i^T\beta_\tau+(1-2\tau)z_i)$and variance $(2m_i)$ truncated to $(-\infty,0)$.

3.3.2-Updating conditional posterior of β is

$\beta|y_i^*,z,\tau,\lambda \sim \text{N}(\mu_\beta,\Lambda_\beta^{-1}),\ \mu_\beta=\Lambda_\beta^{-1}X^TS(y-(1-2\tau)z))=\Lambda_\beta^{-1}\frac{1}{2}D^{-1}(y-(1-2\tau)z)),\ \Lambda_\beta=X^TSX+(\sigma^2A)^{-1}=\frac{1}{2}X^TD^{-1}X+\frac{1}{\sigma^2}A^{-1}$

$,V=diag(Z_1,Z_2,\ldots.Z_n)$

3.3.3-Updating conditional posterior of σ^2 is

$\sigma^2|\beta,\tau_{1:P} \sim Inverse$ gamma with rate paramter $\frac{p}{2}$ and scale paramter $\frac{1}{2}\beta^TA^{-1}\beta$

Where $A=diag(\tau_1,\tau_2,\ldots..\tau_p)$

3.3.4-Updating conditional posterior of τ_j is

$$p(\tau_j|.)\ \propto\ \tau_j^{1/2}\ \exp\left(\frac{-\beta_j^2}{2\sigma^2\tau_j}-\frac{\lambda}{2}\ \tau^2\right)$$

3.3.5-Updating conditional posterior of z_i is

$$p(z_i|y_i^*,\beta)\ \propto\ z_i^{-1/2}\ \exp\left(\frac{(y_i^*-x_i^t\beta_\tau-(1-2\tau)z_i)^2}{4z_i}-\frac{1}{2}\ z_i\right)$$

3.3.6-Updating conditional posterior of λ is

$$\lambda \sim \text{Gamma (c + p, d} + \frac{1}{2}\sum_{j=1}^{p}\tau_j^2$$

The Gibbs sampler described above sequentially draws each unknown parameter, as well as the latent variables $\text{y}_i$and z_i, from their respective full conditional distributions, given all other unknowns. During each iteration, the sampler updates all $(\text{y}_i,\text{z}_i,\beta,\sigma^2,\text{z}_i,\tau_j,\lambda)$. In both simulation studies and real-data applications, the algorithm is executed for 11,000 iterations, with the initial 1,000 iterations discarded as burn-in.

4. Simulation studies

In this section, we conduct a series of simulation experiments to evaluate, performance of the proposed method. The simulations are designed to examine the accuracy, robustness, and computational efficiency of our proposed method under various scenarios. We assess the performance of the proposed Bayesian method with a scale-normal Rayleigh mixing prior distribution (BSCNRMixing) in comparison with several benchmark approaches. Specifically, we include the standard Tobit quantile regression estimator based on Powell's method, implemented through the crq() function in Koenker's framework (denoted as *crq*), the Bayesian adaptive elastic net Tobit quantile regression introduced by Alhamzawi (2014) [2] (referred to as *BAnet*), and the Bayesian Tobit quantile regression model employing new hierarchical Laplace prior distributions, recently proposed by Alhusseini and Jaber (2023) (referred to as *NBTQR*). The competing methods are evaluated using the root mean square error referred to as (RMSE),where RMSE = $\sqrt{\frac{1}{r}\sum_{i=1}^{r}(y_i-\hat{y}_i)^2, where\ r\ number\ of\ iteration}$

The standard deviations are also introduced.

$y = \max(0, y_i^*)$,

$y_i^* = x_{1i} + x_{8i} + \epsilon_i, i = 1,2, \dots \dots .100$

The explanatory variables $(x_{1i}, x_{2i}, x_{3i}, x_{4i}, x_{5i}, x_{6i}, x_{7i}, x_{8i})$, were simulated from an 8-dimensional multivariate normal distribution $X \sim N(\mu, \Sigma), \mu \in R^n$ is the mean vector and Σ is the covariance matrix defined as $(\Sigma_x)_{a,b} = (2^{-1})^{|a-b|}\ where\ a\ is\ row\ and\ b\ is\ column.$

We carry out two simulation studies.

Simulation example 1: $\beta = (1,0,0,0,0,0,0,0)^T$ is very sparse case

Simulation example 2: $\beta = (0.85,0.85,0.85,0.85,0.85,0.85,0.85,0.85)^T$

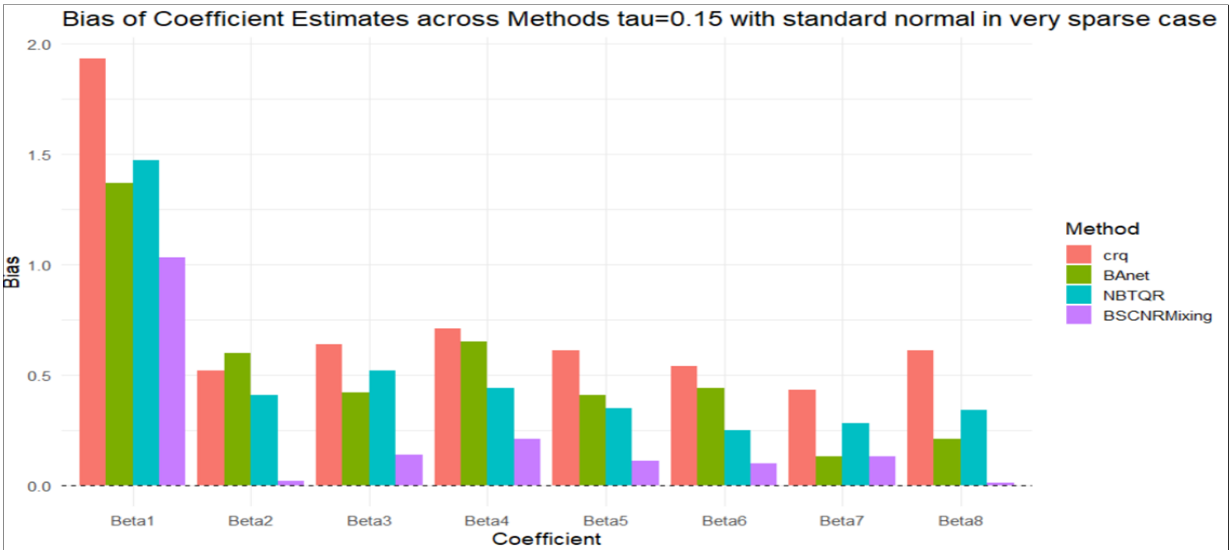
The rows of X follow is $Normal\ with\ mean\ vector\ \underline{0}$ and variance Σ_x, $X \sim N_p(0, \Sigma)$ with $(\Sigma_x)_{r.c} = (0.5)^{|r+c|}$. The random error $(u_i, i = 1,2,3, \dots .n, n = 100)$ are obtained from three separate distributions: $u_i \sim N(0,1)$, the random error u_i is distributed with mean 0 and variance 1. $u_i \sim t(3)$, the random error u_i is distributed as Student's t-distribution with 3 degree of freedom. $u_i \sim Lplace(0,1)$ the random error u_i is distributed as Laplace-distribution with 0,1 location and scale respectively. In this paper, we used five specific quantile levels are $\tau_1 = 0.15, \tau_2 = 0.30, \tau_3 = 0.45, \tau_4 = 0.60, \tau_5 = 0.75\ and\ \tau_6 = 0.90$. For each simulation example.

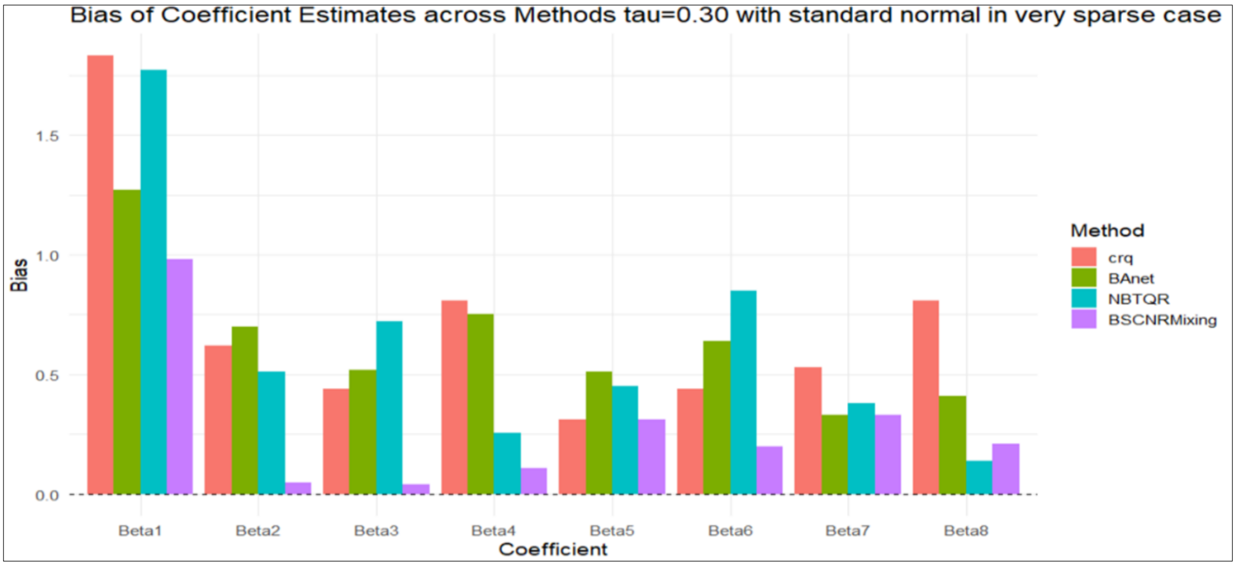
Table 1: Root mean square error(RMSE) and standard division are SD

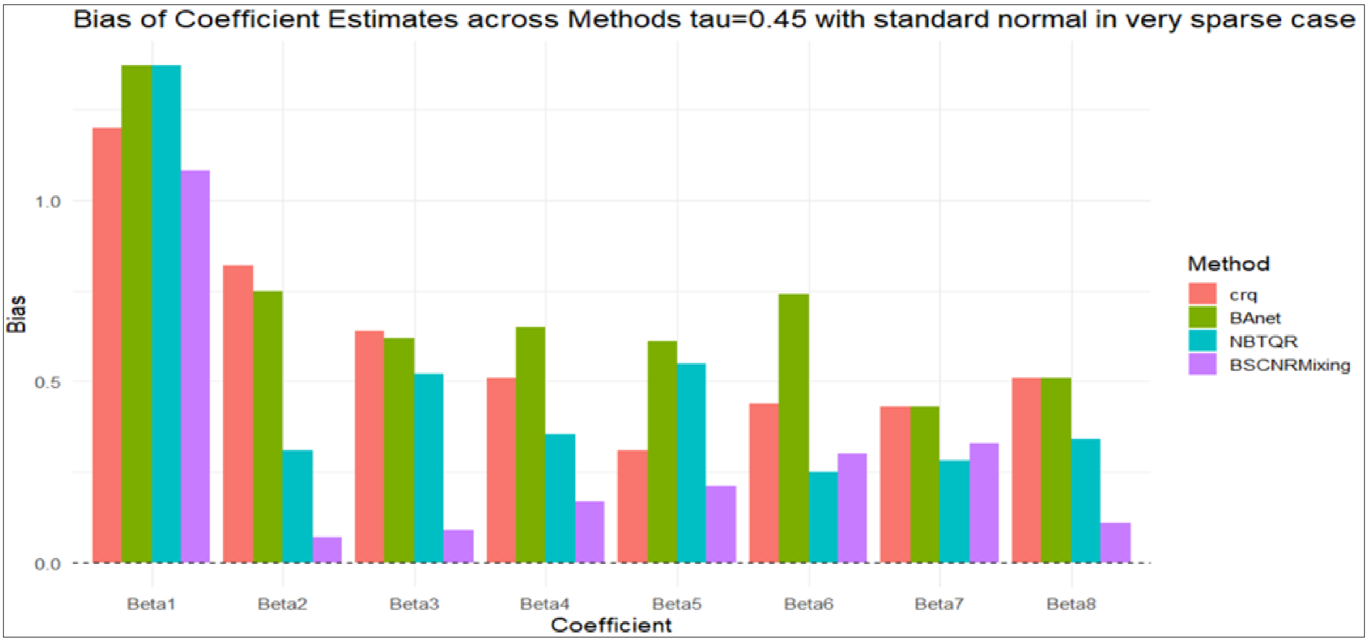
	Quantile level	Methods	$u_i \sim \boldsymbol{N(0,1)}$	$u_i \sim \boldsymbol{t_{(4)}}$	$\epsilon_i \sim \boldsymbol{\chi^2_{(4)}}$
Sim1	$\tau_1 = 0.15$	crq	1.4465 (0.8846)	1.3365 (0.7984)	1.2987 (0.7514)
		BAnet	1.2526 (0.7822)	1.2141 (0.7614)	1.1244 (0.6122)
		NBTQR	0.9645 (0.4854)	0.9548 (0.5864)	0.8275 (0.4168)
		BSCNRMixing	0.5312 (0.3534)	0.6547 (0.2845)	0.5854 (0.3421)
	$\tau_2 = 0.30$	crq	1.3554 (0.8246)	1.3144 (0.9126)	1.2112 (0.7217)
		BAnet	1.2471 (0.8254)	1.2792 (0.8710)	1.2483 (0.7211)
		NBTQR	0.9427 (0.4587)	0.7641 (0.4554)	0.7154 (0.4614)
		BSCNRMixing	0.4647 (0.2095)	0.3754 (0.1637)	0.3777 (0.2041)
	$\tau_3 = 0.45$	crq	1.465 (0.846)	1.472 (0.743	1.257 (0.792)
		BAnet	1.257 (0.783)	1.261 (0.728)	1.362 (0.782)
		NBTQR	0.829 (0.396)	0.694 (0.284)	0.834 (0.356)
		BSCNRMixing	0.405 (0.261)	0.327 (0.328)	0.463 (0.362)
	$\tau_4 = 0.60$	crq	1.471 (0.945)	1.257 (0.792)	1.685 (0.867)
		BAnet	1.515 (0.832)	1.634 (0.957)	1.515 (0.832)
		NBTQR	0.945 (0.539)	0.748 (0.436)	0.892 (0.372)
		BSCNRMixing	0.472 (0.276)	0.361 (0.192)	0.463 (0.362)
	$\tau_5 = 0.75$	crq	1.175 (0.938)	1.019 (0.751)	1.212 (0.792)
		BAnet	1.374 (0.804)	0.828 (0.927)	.007 (0.871)
		NBTQR	0.756 (0.362)	0.871 (0.415)	0.709 (0.361)
		BSCNRMixing	0.518 (0.268)	0.462 (0.142)	0.462 (0.142)
	$\tau_5 = 0.90$	crq	1.251 (0.863)	1.106 (0.856)	0.816 (0.672)
		BAnet	1.262 (0.953)	0.943 (0.905)	0.709 (0.361)
		NBTQR	0.844 (0.264)	0.821 (0.431	0.414 (0.132)
		BSCNRMixing	0.615 (0.252)	0.572 (0.183)	0.356 (0.089)
Sim2	$\tau_1 = 0.15$	crq	1.251 (0.863)	1.464 (0.954)	0.892 (0.411)
		BAnet	0.826 (0.461)	0.745 (0.363)	1.064 (0.761)
		NBTQR	0.854 (0.475)	0.682 (0.201)	0.764 (0.176)
		BSCNRMixing	0.482 (0.184)	0.461 (0.095)	0.381 (0.071)
	$\tau_2 = 0.30$	crq	1.0573 (0.7937)	1.5215 (0.9836)	1.246 (0.943)
		BAnet	0.7839 (0.2393)	0.6582 (0.3204)	0.857 (0.463)

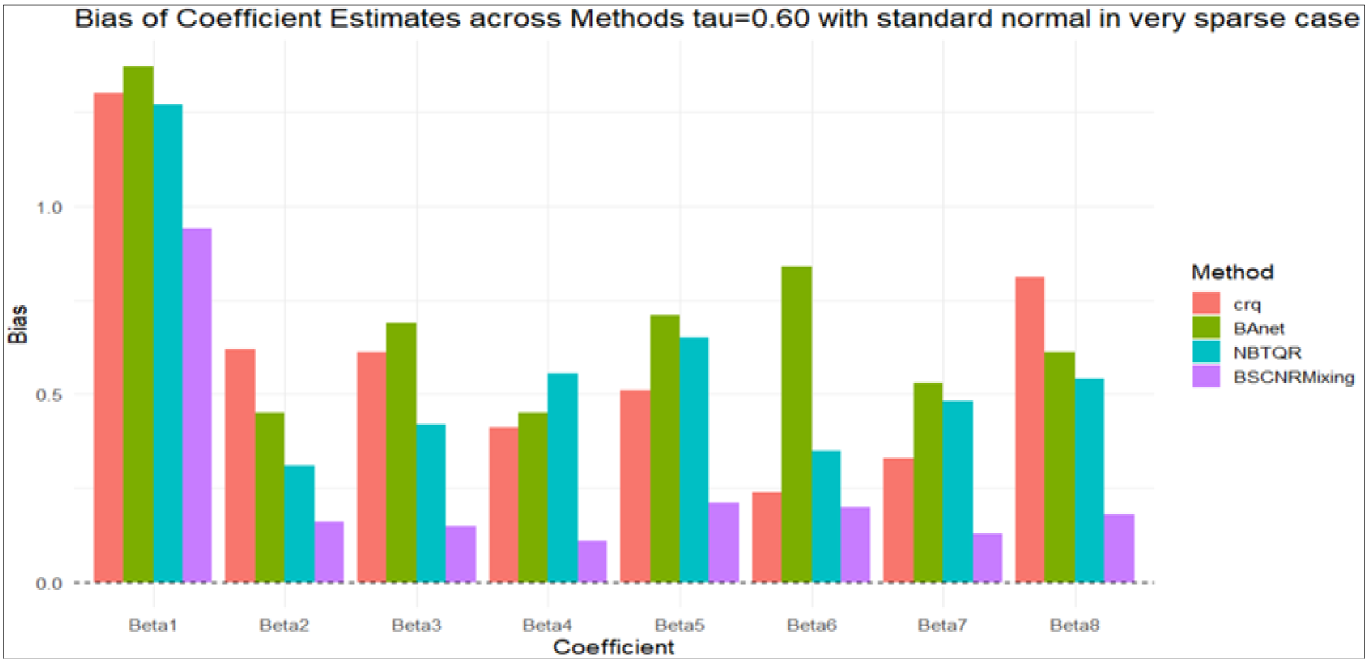
		NBTQR	0.6574 (0.3564)	0.4998 (0.1189)	0.682 (0.217)
		BSCNRMixing	0.3794 (0.1143)	0.3220 (0.1097)	0.263 (0.028)
	$\tau_3 = 0.45$	crq	1.2261 (0.7628)	1.515 (0.832)	1.374 (0.804)
		BAnet	0.7679 (0.4361)	0.945 (0.539)	0.681 (0.361)
		NBTQR	0.8627 (0.4316)	0.863 (0.572)	0.396 (0.094)
		BSCNRMixing	0.5263 (0.3028)	0.372 (0.193)	0.246 (0.096)
	$\tau_4 = 0.60$	crq	0.921 (0.574)	0.964 (0.562)	0.832(0.610)
		BAnet	0.763 (0.430)	0.714(0.505)	0.924(0.601)
		NBTQR	0.581 (0.294)	0.624(0.175)	0.573(0.267)
		BSCNRMixing	0.473 (0.219)	0.562 (0.373)	0.482(0.195)
	$\tau_5 = 0.75$	crq	0. 708 (0.483)	0. 729 (0.509)	0. 708 (0.483)
		BAnet	0.696 (0.523)	0.638 (0.466)	0.518 (0.319)
		NBTQR	0.564 (0.464)	0.515 (0.391)	0.459 (0.253)
		BSCNRMixing	0.487 (0.312)	0.4862 (0.293)	0.375 (0.164)
	$\tau_6 = 0.90$	crq	0.798 (0. 562)	0. 708 (0.483)	0.766 (0.506)
		BAnet	0.696 (0.523)	0.625 (0.451)	0.584 (0.358)
		NBTQR	0.601 (0.497)	0.529 (0.339)	0.4862 (0.293)
		BSCNRMixing	0.509 (0.364)	0.493 (0.106)	0.372 (0.130)

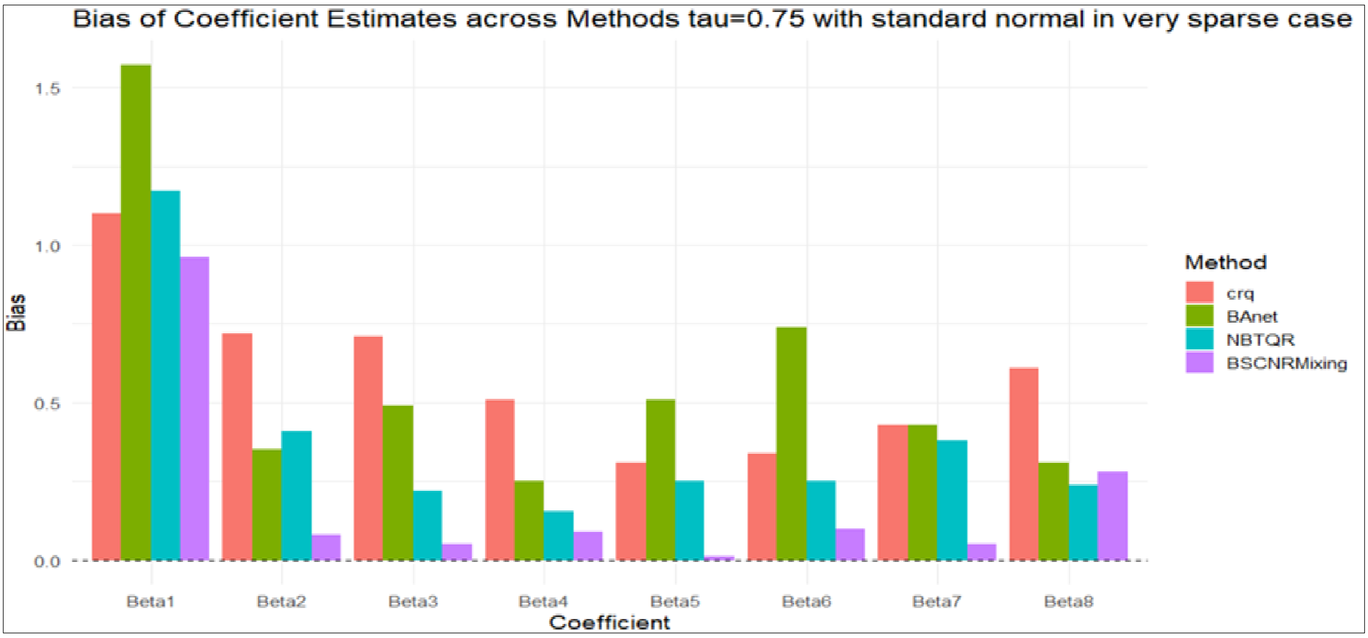
Table 1 summarizes the root mean square error (RMSE) and standard deviation (SD) obtained from 100 simulated datasets, providing a comparative evaluation of the four methods under consideration. t is evident from Table 1 that the proposed BSCNRMixing method demonstrates superior performance relative to the crq, BAnet, and NBTQR methods. This is apparent The RMSE obtained from our proposed method(BSCNRMixing) is much smaller than RMSR that produced by the crq, BAnet, and NBTQR methods across all distributions under consideration. Moreover, throughout the simulation studies involving different error distributions. Also, The standard deviation (SD) produced by the proposed method is considerably smaller than that of the competing methods (crq, BAnet, and NBTQR). Instead of looking at the RMSE and the SDs, one may also look at the bias of parameters estimation. The figure -1- lists the bias of parameters estimation in the first simulation Instead of looking at the RMSE and the SDs, one may also look at the bias of parameters estimation. The figure -1- lists the bias of parameters estimation In the first simulation, when the random error follows the standard normal distribution.






Bias of Coefficient Estimates across Methods tau=0.45 with standard normal in very sparse case
Bias
1.0
0.5
0.0
Beta1
Beta2
Beta3
Beta4
Beta5
Beta6
Beta7
Beta8
Coefficient
Method
crq
BAnet
NBTQR
BSCNRMixing


Bias of Coefficient Estimates across Methods tau=0.60 with standard normal in very sparse case
Bias
1.0
0.5
0.0
Beta1
Beta2
Beta3
Beta4
Beta5
Beta6
Beta7
Beta8
Coefficient
Method
crq
BAnet
NBTQR
BSCNRMixing


Bias of Coefficient Estimates across Methods tau=0.75 with standard normal in very sparse case
Bias
1.5
1.0
0.5
0.0
Beta1
Beta2
Beta3
Beta4
Beta5
Beta6
Beta7
Beta8
Coefficient
Method
crq
BAnet
NBTQR
BSCNRMixing

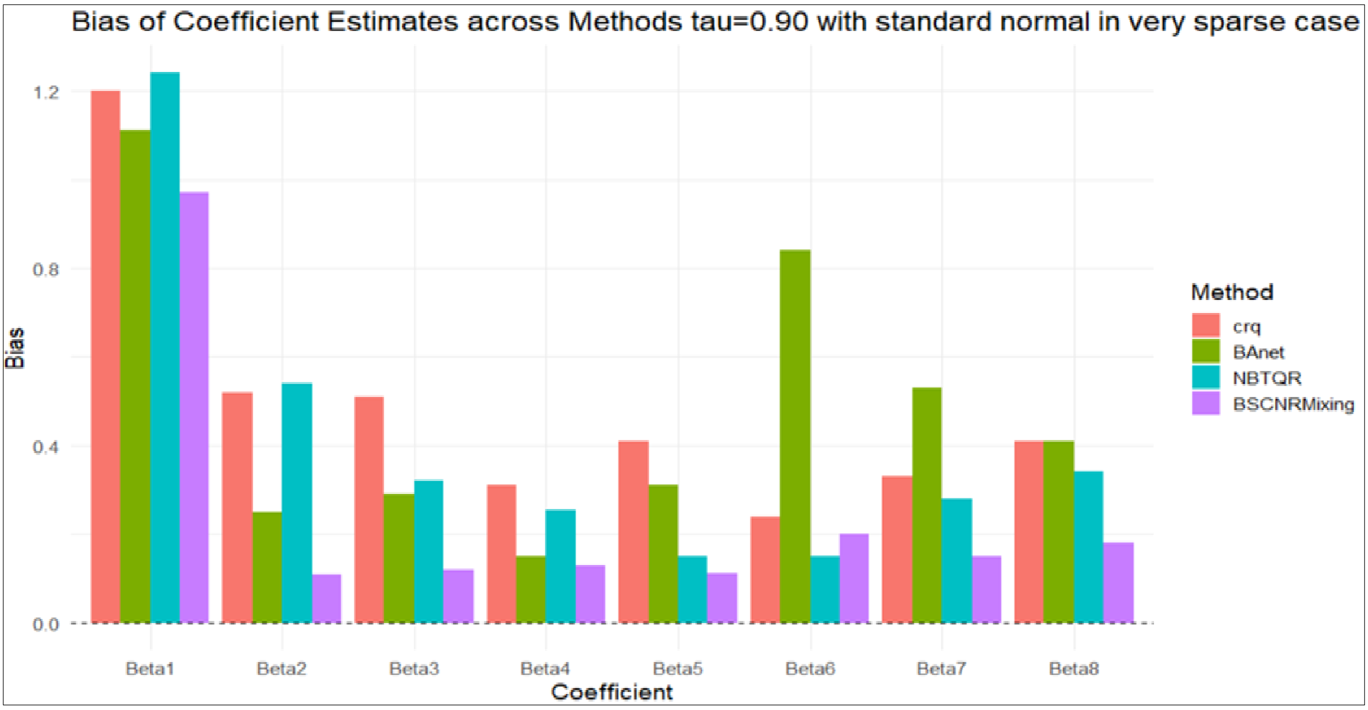


Fig 1: is illustrates the bias of the estimates across the different methods. It is evident that the BSCNRMixing method achieves the lowest bias. Consequently, the our proposed method is a good overall performance compared by other three methods.

5. Real data

To demonstrate the applicability of the our proposed method (BSCNRMixing) and to conduct a comparative assessment against existing other method. The focus will be on Household Survey data, that collected from Household Survey of Diwaniyah Governorate. The dataset comprises n=565 observations, of which 275 are censored. This corresponds to a censoring proportion of approximately 48.67%. The response variable in this study is defined as the monthly health expenditure (MHE) incurred by each family and eleven independent variables are Monthly household income (MHI) denoted as x_1, Number of chronic in family members (N C F M) denoted as x_2, Household head's age or mean age of adults (H HAMAA) denoted as x_3, Education level of the Household head(ELHH) denoted as x_4, Distance from the household to the nearest health center (DHNHC) denoted as x_5, Household Size denoted as x_6, Household includes an infant or a pregnant member (HIIPM) denoted as x_7, Health Awareness Score (HAS) denoted as x_8, Number of children below five years in one household (NCBF), denoted as x_9, Is the household covered by health insurance (HCHI) denoted as x_{10}, Engagement in healthy dietary habits (EHDH)denoted as x_{11}. In Section 5, we set $a = b = c = d = 0.1$ and ran the our algorithm for 11,000 iterations, discarding the first 1,000 as burn-in.

Table 2: Mean squared errors (MSE) and standard deviation (SD) for the methods under comparison.

Methods	$\tau_1 = 0.15$	$\tau_2 = 0.30$	$\tau_3 = 0.45$	$\tau_4 = 0.60$	$\tau_5 = 0.75$	$\tau_6 = 0.90$
crq	0.554 (0.214)	0.531 (0.120)	0.528 (0.151)	0.538 (0.127)	0.497 (0.274)	0.526 (0.305)
BAnet	0.412 (0. 238)	0.452 (0. 284)	0.454 (0. 214)	0.454 (0. 239)	0.459 (0. 234)	0.487 (0. 183)
NBTQR	0.385 (0.208)	0.363 (0.127)	0.318 (0.206)	0.301 (0.109)	0.311 (0.234)	0.376 (0.204)
BSCNRMixing	0.225 (0.102)	0.246 (0.114)	0.230 (0.118)	0.254 (0.110)	0.224 (0.117)	0.266 (0.112)

As well, Table 2 shows that the less MSE is in the our proposed method (BSCNRMixing), Therefore, our proposed method (BSCNRMixing) demonstrates superior performance in variable selection and parameter estimation at tobit quantile regression, even with real data, compared to the three other methods.

Table 3: coefficient estimation for the four methods in the comparison.

variables	Variables symbols	BSCNRMixing					
	Quantile level	$\tau_1 = 0.15$	$\tau_2 = 0.30$	$\tau_3 = 0.45$	$\tau_4 = 0.60$	$\tau_5 = 0.75$	$\tau_6 = 0.90$
x_1	MHI	0.1842	0.1841	0.2107	0.1163	0.2821	0.1152
x_2	N C F M	1.1275	1.0041	1.0002	1.1183	1.3164	1.1247
x_3	H HAMAA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
x_4	ELHH	0.0062	0.0051	0.0025	0.0003	0.0082	0.0090
x_5	DHNHC	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
x_6	HIIPM	1.0765	1.4212	1.1109	1.1193	1.1002	1.0218
x_7	HAS)	0.0786	0.0245	0.1285	0.1346	0.1809	0.2370
x_8	NCBF	-0.1527	-0.0238	-0. 0331	-0.1019	-0.2316	-0.1016
x_9	HCHI	0.0091	0.0011	0.0007	0.0000	0.0000	0.0000
x_{10}	EHDH	0.0021	0.0132	0.0103	0.0016	0.0182	0.0274
x_{11}	MHI	0.0004	0.0001	0.2117	0.0255	0.0105	0.0018

The results presented in Table 3 represent the estimated coefficients of the Tobit quantile regression model obtained using our proposed method across all quantile levels. It can be observed that some estimated coefficients are exactly zero, while others take nonzero values. These results indicate that our method successfully achieves the intended objectives of both parameter estimation and variable selection.

The trace plots in Figure -2- demonstrate that the MCMC samples converge to stationarity, indicating good mixing of the Markov chains and convergence to the target distribution.

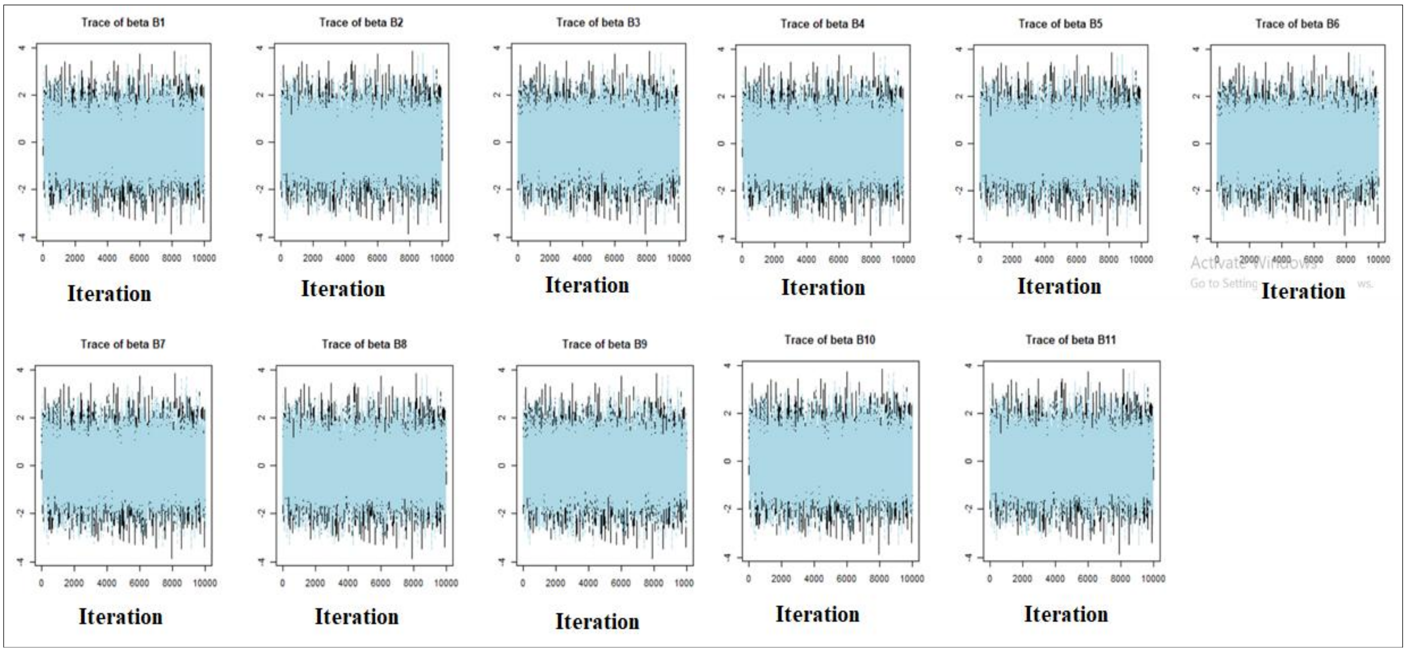


Fig 2: Show the Trace Plots of $\beta_1 - \beta_{11}$ of the conditional Posterior parameter estimates

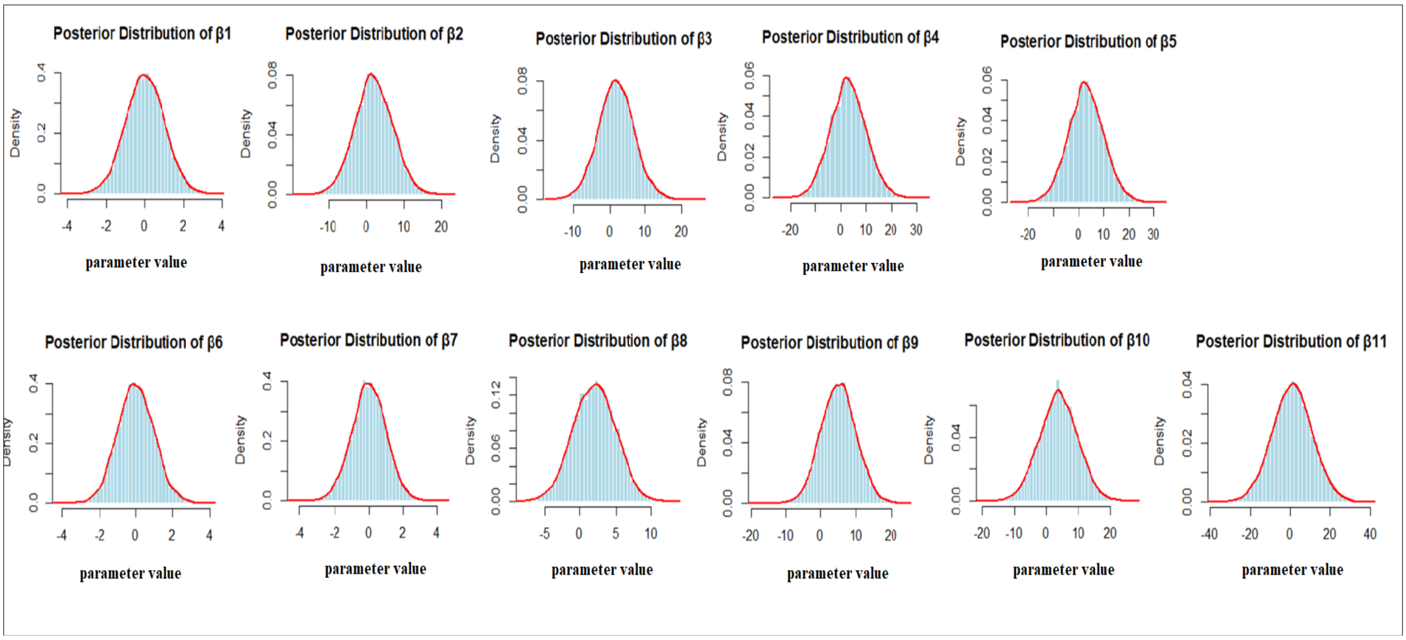


Fig 3: Show the histograms of parameter estimates $\beta_1 - \beta_{11}$.

6. Conclusion and Future work

This study proposed a novel Bayesian Lasso framework for Tobit Quantile Regression by employing a Scale Mixture of Normals with Rayleigh priors (BSCNRMIXING prior). The hierarchical representation allows for efficient posterior computation via Gibbs sampling and provides a flexible approach to simultaneous coefficient estimation and variable selection under censoring. Simulation studies across different quantile levels and error distributions demonstrated that the proposed method (BSCNRMixing) consistently outperformed existing approaches (crq, BAnet, NBTQR) in terms of root mean square error (RMSE), standard deviation (SD), and bias of parameter estimates. The superior performance was particularly evident in high-sparsity scenarios, where variable selection is crucial, as well as in heavy-tailed error distributions where robustness is essential. The application to real household health expenditure data further confirmed the effectiveness of the proposed approach. Compared with competing Bayesian and frequentist estimators, the BSCNRMixing prior-based method achieved lower mean squared errors (MSE) and demonstrated greater stability in coefficient estimation across different quantile levels. Importantly, the proposed framework successfully identified influential covariates, while shrinking irrelevant ones toward zero, thereby enhancing interpretability and predictive accuracy. Our proposed method can be extended in several directions. For instance, it may be adapted to the Bayesian elastic net Tobit quantile regression model by employing a scale mixture of normal distributions with a Rayleigh mixing distribution.

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